A STUDY ON A SPECIAL CASE OF SENSITIVITY OF BAYES ESTIMATES OF WEIGHTED WEIBULL LENGTH BIASED DISTRIBUTION WITH A CHANGE POINT MODEL USING ASYMMETRIC LOSS FUNCTIONS

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Abstract:

In this research paper, we have made an attempt to study a special case of length biasedness of Weibull Life Time Model where the Weibull Length Biased Model gets transformed into Weighted Weibull Length Biased Distribution, where a change point model is developed related to the Weibull Length Biased Life Time Model. We have obtained the posterior densities of $\theta_1, \theta_2, \beta$ and ‘m’ for this model. We have also obtained the Bayes Estimates of $\theta_1, \theta_2, \beta$ and ‘m’ under asymmetric loss functions.
studied the sensitivity of Bayes Estimates with respect to change in the prior of the parameters for the life time model under study. Finally, we have given the conclusions on the basis of our study.

**Key Words:**
Bayes Estimates, Change Point, Life Time Model, Asymmetric Loss Functions, Length Biasedness, Weighted Weibull Length Biased Distribution.

1. **INTRODUCTION:**

The principle of length biasedness and the length biased distribution was studied by Gupta and Akman in 1995. On the basis of their study, an article was developed and presented on the study of human families and wildlife populations. It presented a list of the most common forms of the weight function useful in scientific and statistical literature. Later on, it was studied by Patill and Rao in 1978 and developed further in 1986. It was in the year 1978 that Patill and Rao gave a table for some basic distributions and their length biased forms such as Beta, Gamma, Lognormal and Pareto distributions. The weighted version of the bivariate three-parameter logarithmic series distribution was studied by Gupta and Tripathi in 1996. It has wide applications in many fields such as ecology, social and behavioural sciences and species abundance studies. Further studies confirmed that there were some basic theorems for weighted distributions and size-biased as special case and on the basis of that study, the conclusion was made that the length-biased version of some mixture of discrete distributions arises as a mixture of the length-biased version of these distributions. A significant work was done to characterize relationships between original distributions and their length biased versions and therefore it became necessary to work further on this aspect. The length-biased distribution has very wide applications in biomedical area such as family history and disease, survival and intermediate events and latency period of AIDS due to blood transfusion.

Let us assume that X be a random variable following the Weibull distribution with pdf as under:

\[ g(x) = \beta \theta x^{\beta-1} \exp (-\theta x^{\beta}) \]

where \( x \geq 0, \beta > 0, \theta > 0 \)

(1)

Here \( \beta \) is the shape parameter and \( \theta \) scale parameter. We know that Weibull distribution is very flexible
and this is due to its application in modeling in both the cases, viz. increasing ($\beta > 1$) as well as decreasing ($\beta < 1$) failure rates.

Moreover, we have $E(X) = \Gamma \beta^{-1} / \beta \theta^{\beta^{-1}}$.

Let $T$ be a non negative random variable, $T$ is said to have the Weibull length-biased distribution it will be abbreviated as WLB if its density function is given by:

$$f(t) = \frac{\beta^2 \theta^{\frac{1}{\beta} + 1}}{\Gamma \frac{1}{\beta}} t^\beta e^{-\theta t^\beta}$$

where $\beta, \theta > 0$ and $t > 0$ (2)

The density (2) can be obtained by combining the definition of the length-biased distribution given by:

$$f(t) = \frac{tg(t)}{E(t)}$$

It can be explained as follows:

Suppose that the lifetime of a given sample of items follows Weibull Distribution and the density of the original distribution given in (2). As per the property of this distribution, the item doesn’t have the same chance of being selected but each one is selected according to its length or life length then the resulting distribution is neither Weibull nor Weibull Length Biased. The resultant distribution becomes the special case which we term as Weighted Weibull Length-Biased Distribution.

The reliability function is given by:

$$R(t) = 1 - \frac{\beta \gamma(\frac{1}{\beta} + 1, \theta t^\beta)}{\Gamma \frac{1}{\beta}}$$

Here, we note that the numerator represents the incomplete gamma function as:

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$$

(5)
2. CHANGE POINT MODEL UNDER STUDY:

Let \( T_1, T, T_3, \ldots, T_n \) (n \( \geq 3 \)) be a sequence of observed life time data. Let first ‘m’ observations \( T_1, T_2, \ldots, T_m \) have come from the Weighted Weibull Length Biased distribution with probability density function \( WWLB(\beta, \theta_1) \).

\[
f(t_i) = \beta \theta_1^2 t_i^{2\beta - 1} e^{-\theta_1 t_i^\beta} \quad \text{where } i=1,2,\ldots,m
\]

and later \( (n-m) \) observations coming from the Weighted Weibull Length Biased with probability density function \( WWLB(\beta, \theta_2) \)

\[
f(t_i) = \beta \theta_2^2 t_i^{2\beta - 1} e^{-\theta_2 t_i^\beta} \quad \text{where } i=m+1,\ldots,n
\]

where \( \beta, \theta_1, \theta_2 > 0 \)

The likelihood function, given the sample information \( \mathcal{T} = (T_1, T_2, \ldots, T_m, T_{m+1}, \ldots, T_n) \) is

\[
L(\theta_1, \theta_2, \beta | \mathcal{T}) = \beta^n \theta_1^{2m} A_1^{2\beta - 1} e^{-\theta_1 A_2^{2(n-m)}} e^{-\theta_2 A_3}
\]

where

\[
A_1 = \prod_{i=1}^{n} t_i
\]

\[
A_2 = A_2(m, \beta / t_i) = \sum_{i=1}^{m} t_i^\beta
\]

\[
A_3 = A_3(m, \beta / t_i) = \sum_{i=m+1}^{n} t_i^\beta
\]

3. USE OF INFORMATIVE PRIOR IN POSTERIOR DISTRIBUTION FUNCTIONS:

Let us assume that the marginal prior distribution of ‘m’ be following discrete uniform distribution over the set \( \{1, 2, \ldots, n-1\} \) citing the research work of Broemeling et al.(1987).

So we take \( g(m) = \frac{1}{n-1} \)

We cite the research work of Calabria and Pulcini carried out in 1992 and suppose the marginal prior distribution on \( \beta \) to be uniform over the interval \( \beta_1, \beta_2 \) as under:
\[ g(\beta) = \frac{1}{\beta_2 - \beta_1} \quad \beta_1 \leq \beta \leq \beta_2 \quad (10) \]

Further we also cite the phenomenal research work of N. Sanjari Farsipour and H. Zakerzadeh done in the year 2005. As per their work, we assume that the scale parameters \( \theta_1 \) and \( \theta_2 \) are unknown and we take the Inverted Gamma prior with probability density functions with respective means values \( \mu_1, \mu_2 \) and common standard deviation \( \sigma \) as under:

\[
g\left( \frac{\theta_i}{\beta} \right) = \frac{a_i \beta}{\Gamma \beta} \theta_i^{-\beta+1} e^{-a_i/\theta_i} \quad i = 1, 2 \quad (11)\]

It is quite clear that this prior distribution has significant advantages over many other distributions because of its analytical tractability, richness and interpretability.

Let the prior information be given in terms of the prior means \( \mu_1, \mu_2 \) and variances \( \sigma_1^2, \sigma_2^2 \). Then \( \mu_i = E[\theta_i] = \frac{a_i}{\beta-1} \) and \( \sigma_i^2 = \frac{a_i^2}{(\beta-1)^2(\beta-2)} \) which gives

\[
a_i = \mu_i \left( \frac{\mu_i^2}{\sigma_i^2} + 1 \right) \quad \text{and} \quad \beta = 2 + \left( \frac{\mu_i^2}{\sigma_i^2} \right) \quad \text{where} \quad i = 1, 2 \quad (12)\]

Thus if we have prior knowledge of \( \mu_1, \mu_2 \) and \( \sigma_1^2, \sigma_2^2 \) then the Inverted gamma parameters \( a_i, \beta \), where \( i = 1, 2 \) can be obtained from (12).

We assume that \( \theta_1, \theta_2, \beta \) and \( m \) are priori independent. The joint prior density will be:

\[
g(\theta_1, \theta_2, \beta, m) = \frac{1}{n-1} \frac{1}{\beta_2 - \beta_1} \frac{a_i \beta}{\Gamma \beta} \theta_1^{-(\beta+1)} e^{-a_1/\theta_1} \frac{a_2 \beta}{\Gamma \beta} \theta_2^{-(\beta+1)} e^{-a_2/\theta_2} = K_1 \frac{a_i \beta}{\Gamma \beta} \theta_1^{-(\beta+1)} e^{-a_1/\theta_1} \frac{a_2 \beta}{\Gamma \beta} \theta_2^{-(\beta+1)} e^{-a_2/\theta_2} \quad (13)\]

where \( K_1 = \frac{1}{n-1} \frac{1}{\beta_2 - \beta_1} \)
The joint posterior density of parameters \( \theta_1, \theta_2, \beta, \) and \( m \) is obtained using the likelihood function (7) and the joint prior density of the parameters in (13) as under:

\[
g(\theta_1, \theta_2, \beta, m \mid T) = \frac{L(\theta_1, \theta_2, \beta, m\mid T)g(\theta_1, \theta_2, \beta, m)}{h(t)}
\]

\[
= \beta^n \theta_1^{2m} A_1 2^\beta -1 e^{-\theta_1 A_2 \theta_2 2(n-m)} e^{-\theta_2 A_3} K_1 \Gamma_{\beta} \theta_1^{-(\beta+1)} e^{-a_1/\theta_1} \theta_1^{a_1/\theta_1} \theta_2^{-(\beta+1)} e^{-a_2/\theta_2} h^{-1}(T)
\]

\[
= K_1 \frac{\Gamma_{\beta}}{\Gamma_{\beta}} A_1 2^\beta -1 \beta^n \theta_1^{2m -\beta -1} e^{-(\theta_1 A_2 + a_1/\theta_1)} \theta_2^{2(n-m) -\beta -1} e^{-(\theta_2 A_3 + a_2/\theta_2)} h^{-1}(T)
\]

(14)

Here, \( h(T) \) is the marginal posterior density of \( T \).

\[
h(T) = \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \int_0^\infty L(\theta_1, \theta_2, \beta, m\mid T)g(\theta_1, \theta_2, \beta, m) \text{d}\theta_1 \text{d}\theta_2 \text{d}\beta
\]

\[
= \sum_{m=1}^{n-1} K_1 \int_{\beta_1}^{\beta_2} \frac{\Gamma_{\beta}}{\Gamma_{\beta}} A_1 2^\beta -1 \beta^n \{ \int_0^\infty \theta_1^{2m -\beta -1} e^{-(\theta_1 A_2 + a_1/\theta_1)} \text{d}\theta_1 \}
\]

\[
= \sum_{m=1}^{n-1} K_1 \int_{\beta_1}^{\beta_2} \frac{\Gamma_{\beta}}{\Gamma_{\beta}} A_1 2^\beta -1 \beta^n l_1(m, \beta) l_2(m, \beta) \text{d}\beta
\]

(15)

where \( l_1(m, \beta) = \int_0^\infty \theta_1^{2m -\beta -1} e^{-(\theta_1 A_2 + a_1/\theta_1)} \text{d}\theta_1 \)

\[
= 2A_2^{-[2m-\beta]/2} \left[ \frac{1}{a_1} \right]^{-[2m-\beta]/2} \text{Bessel K} \left[ -2m + \beta, 2\sqrt{a_1} \sqrt{A_2} \right]
\]

(16)

\( l_2(m, \beta) = \int_0^\infty \theta_2^{2(n-m) -\beta -1} e^{-(\theta_2 A_3 + a_2/\theta_2)} \text{d}\theta_2 \)
\[
2A_3^{-[2(n-m)-\beta]/2} \left[ \frac{1}{a_2} \right]^{-[2(n-m)-\beta]/2} \text{Bessel K} \left( -2(n-m) + \beta, 2\sqrt{a_2\sqrt{A_3}} \right)
\]

(17)

where \text{Bessel K} \left( -2m + \beta, 2\sqrt{a_1\sqrt{A_2}} \right) \text{ and Bessel K} \left( -2(n-m) + \beta, 2\sqrt{a_2\sqrt{A_3}} \right) \text{ are defined as}

\[
\int_0^\infty b^{1-m} e^{-(bc+d/b)} dB = 2(c/d)^{m/2} \text{Bessel K}[m, 2\sqrt{c}\sqrt{d}]
\]

(18)

The marginal density of \( \theta_1 \) say \( g(\theta_1|T) \) is as,

\[
g(\theta_1|T) = K_1 \sum_{m=1}^{n-1} \int_0^{\beta_2} \int_0^{\infty} g(\theta_1, \theta_2, \beta|T) d\theta_2 d\beta = K_1 \sum_{m=1}^{n-1} \theta_1^{2m-\beta-1} e^{-(\theta_1 \alpha_2^2 + \frac{\alpha_1}{\theta_1})} 2A_3^{-[2(n-m)-\beta]/2}
\]

\[
\left[ \frac{1}{a_2} \right]^{-[2(n-m)-\beta]/2} \text{Bessel K} \left( -2(n-m) + \beta, 2\sqrt{a_2\sqrt{A_3}} \right) \int_{\beta_1}^{\beta_2} \frac{\alpha_1}{\Gamma(\beta_1)} \frac{\alpha_2}{\Gamma(\beta_2)} A_1^{2\beta-1} \beta^n d\beta h^{-1}(T)
\]

(19)

where \text{Bessel K} \left( -2(n-m) + \beta, 2\sqrt{a_2\sqrt{A_3}} \right) \text{ is same as in (18)}.

The marginal density of \( \theta_2 \) say \( g(\theta_2|T) \) is as,

\[
g(\theta_2|T) = K_1 \sum_{m=1}^{n-1} \int_0^{\beta_2} \int_0^{\infty} g(\theta_1, \theta_2, \beta|T) d\theta_1 d\beta
\]

\[
= K_1 \sum_{m=1}^{n-1} \theta_2^{2(n-m)-\beta-1} e^{-(\theta_2 \alpha_3^2 + \frac{\alpha_2}{\theta_2})} 2A_2^{-[2m-\beta]/2}
\]

\[
\left[ \frac{1}{a_1} \right]^{-[2m-\beta]/2} \text{Bessel K} \left( -2m + \beta, 2\sqrt{a_1\sqrt{A_2}} \right) \int_{\beta_1}^{\beta_2} \frac{\alpha_1}{\Gamma(\beta_1)} \frac{\alpha_2}{\Gamma(\beta_2)} A_1^{2\beta-1} \beta^n d\beta h^{-1}(T)
\]

(20)
where Bessel $K[-2m + \beta, 2\sqrt{a_1A_2}]$ is same as in (18).

The marginal density of $\beta$, say $g(\beta|T)$ will be as under:

$$
g(\beta|T) = K_1 \sum_{m=1}^{n-1} \int_0^\infty \int_0^\infty g(\theta_1, \theta_2, \beta|T) \, d\theta_1 \, d\theta_2
$$

$$
= K_1 \sum_{m=1}^{n-1} 2A_2^{-[2m-\beta]/2} \left[ \frac{1}{a_1} \right]^{-[2m-\beta]/2} \text{Bessel } K[-2m + \beta, 2\sqrt{a_1A_2}] 2A_3^{-[2(n-m)-\beta]/2}
$$

$$
\left[ \frac{1}{a_2} \right]^{-[2(n-m)-\beta]/2} \text{Bessel } K[-2(n-m) + \beta, 2\sqrt{a_2A_3}] \frac{a_1^\beta a_2^\beta}{\Gamma^\beta} A_1^{2\beta-1} \beta^n \, h^{-1}(T)
$$

where Bessel $K[-2m + \beta, 2\sqrt{a_1A_2}]$ and Bessel $K[-2(n-m) + \beta, 2\sqrt{a_2A_3}]$ are same as in (18).

Marginal posterior density of $m$ say, $g(m|T)$ is taken as under:

$$
g(m|T) = K_1 I_3(m) \, h^{-1}(T)
$$

$$
= I_3(m)/\sum_{m=1}^{n-1} I_3(m)
$$

where $I_3(m) = \int_0^\infty \frac{a_1^\beta a_2^\beta}{\Gamma^\beta} A_1^{2\beta-1} \beta^n I_1(m, \beta)I_2(m, \beta) \, d\beta$

where $I_1(m, \beta), I_2(m, \beta)$ are same as in (16) and (17) respectively.
4. USING INFORMATIVE PRIOR AND BAYES ESTIMATES UNDER ASYMMETRIC LOSS FUNCTION:

In this section, we have obtained Bayes Estimates of the change point and parameters $\theta_1$ and $\theta_2$. Here we have used a very useful asymmetric loss function known as the Linex Loss Function. It was introduced by Varian in 1975.

Minimizing the posterior expectation of the Linex loss function $E_m [L_4 (m, d)]$, where $E_m [L_4 (m, d)]$ denotes the expectation of $L_4 (m, d)$ with respect to posterior density $g(m | T)$. We get the Bayes estimate of ‘m’ by means of the nearest integer value, say $m^*_L$, using Linex Loss Function as under:

$$m^*_L = -\frac{1}{q_1} \ln[E(e^{-mq_1})]$$

where $I_3(m)$ same as in (23).

Minimizing expected loss function $E_{\theta_1} [L_4 (\theta_1, d)]$ and using posterior distribution (19) and we get the Bayes Estimates of $\theta_1$, using Linex loss function as

$$\theta^*_1 = -\frac{1}{q_1} \ln[E(e^{-\theta_1 q_1})]$$

$$= -\frac{1}{q_1} \ln[K_1 \sum_{m=1}^{n-1} \int_0^{\infty} [\theta_1 e^{2m-\beta-1} e^{-(\theta_1 A_2 + a_1/\theta_1 + \theta_1 q_1)} d\theta_1] 2A_3^{-[2(n-m)-\beta]/2}]$$

$$Bessel K[-2(n-m) + \beta, 2\sqrt{a_2}\sqrt{A_3}]$$

$$\int_{\beta_1}^{\beta_2} A_1^{2\beta-1} \beta^n d\beta h^{-1}(T)$$
\[ = - \frac{1}{q_1} \ln [K_1 \sum_{m=1}^{\infty} 2 \{ A_2 + q_1 \}^{-(2m-\beta)/2} \left( \frac{1}{a_1} \right)^{-(2m-\beta)/2} \]

\[ \text{Bessel K} \left[ -2m + \beta, 2\sqrt{a_1} \sqrt{A_2 + q_1} \right] \]

\[ 2A_3^{-(2(n-m)-\beta)/2} \left( \frac{1}{a_2} \right)^{-(2(n-m)-\beta)/2} \text{Bessel K} \left[ -2(n - m) + \beta, 2\sqrt{a_2} \sqrt{A_3} \right] \]

\[ \int_{\beta_1}^{\beta_2} \frac{a_1^\beta a_2^\beta}{\Gamma^\beta} A_1 2^{\beta-1} \beta^n \, d\beta h^{-1}(T) \]

(22)

where Bessel K\left[ -2m + \beta, 2\sqrt{a_1} \sqrt{A_2 + q_1} \right] and Bessel K\left[ -2(n - m) + \beta, 2\sqrt{a_2} \sqrt{A_3} \right] are same as is in (18)

Minimizing expected loss function \( E_{\theta_1} [L_4 (\theta_2, d)] \) and using posterior distribution (20), we get the Bayes Estimates of \( \theta_2 \), using Linex Loss Function as

\[ \theta_{2L}^* = - \frac{1}{q_1} \ln [E(e^{-\theta_2 q_1})] \]

\[ = - \frac{1}{q_1} \ln \left[ \int_0^{\infty} g_1(\theta_2 | X), e^{-\theta_2 q_1} \, d\theta_2 \right] \]

\[ = - \frac{1}{q_1} \ln \left[ K_1 \sum_{m=1}^{\infty} \int_0^{\infty} \theta_2^{2(n-m)-\beta} e^{-\theta_2 A_3 + \frac{a_2}{\theta_2}} e^{-\theta_2 q_1} \, d\theta_2 \right] 2A_2^{-(2m-\beta)/2} \left( \frac{1}{a_1} \right)^{-(2m-\beta)/2} \text{Bessel K} \left[ -2m + \beta, 2\sqrt{a_1} \sqrt{A_2} \right] \]

\[ \int_{\beta_1}^{\beta_2} \frac{a_1^\beta a_2^\beta}{\Gamma^\beta} A_1 2^{\beta-1} \beta^n \, d\beta h^{-1}(T) \]

\[ = - \frac{1}{q_1} \ln \left[ K_1 \sum_{m=1}^{\infty} 2 \{ A_3 + q_1 \}^{-(2(n-m)-\beta)/2} \right. \]

\[ \left. \left( \frac{1}{a_2} \right)^{-(2(n-m)-\beta)/2} \text{Bessel K} \left[ -2(n - m) + \beta, 2\sqrt{a_2} \sqrt{A_3 + q_1} \right] \right] \]

\[ 2A_2^{-(2m-\beta)/2} \left( \frac{1}{a_1} \right)^{-(2m-\beta)/2} \text{Bessel K} \left[ -2m + \beta, 2\sqrt{a_1} \sqrt{A_2} \right] \]
\[ \int_{\beta_1}^{\beta_2} \frac{a_1^\beta a_2^\beta}{\Gamma^\beta} A_1^{-1} \beta^{-1} \beta^n \ d\beta h^{-1}(T) \]  

(29)

where Bessel \( K[-2m + \beta, 2\sqrt{a_1\sqrt{A_2}}] \) and

Bessel \( K[-2(n - m) + \beta, 2\sqrt{a_2\sqrt{A_3 + q_1}}] \) are same as is in (18)

Minimizing expected loss function \( E_{\beta}[L_4 (\beta, d)] \) and using posterior distribution (21), we get the Bayes Estimates of \( \beta \), using Linex Loss Function as,

\[ \beta^*_L = -\frac{1}{q_1} \ln[E(e^{-p_1/q_1})] \]

\[ = -\frac{1}{q_1} \ln[\int_{\beta_1}^{\beta_2} g_1(\beta|X) e^{-\beta q_1} d\beta] \]

\[ = -\frac{1}{q_1} \ln[K \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_1^\beta a_2^\beta}{\Gamma^\beta} A_1^{-1} \beta^{-1} \beta^n e^{-\beta q_1} d\beta h^{-1}(T) ] \]

(30)

The Bayes estimate \( m^*_E \) of ‘m’ using General Entropy Loss Function is explained below. It was proposed by Calabria and Pulcini in 1994.

Minimizing the expectation \( [E_m[L_5 (m, d)] \) and using posterior distribution, we get the Bayes estimate ‘m’ by means of the nearest integer value say \( m^*_E \), using General Entropy Loss Function as under:

\[ m^*_E = E(m^{-q_3})^{1/q_3} \]

\[ = \left[ \frac{\sum_{m=1}^{n-1} m^{-q_3} I_3(m)}{\sum_{m=1}^{n-1} I_3(m)} \right]^{1/q_3} \]

(31)

where \( I_3(m) \) is same as in (23).

Further, minimizing the expectation \( [E_{\theta_1} [L_5 (\theta_1, d)] \) and using posterior distribution (19), we get Bayes estimate of \( \theta_1 \) using General Entropy Loss Function as,

\[ \theta^*_E = E(\theta_1^{-q_3})^{1/q_3} \]
\[
= [K] \sum_{m=1}^{n-l} \int_{0}^{\infty} \left[ \theta_1^{2m-\beta-q_3-1} e^{-\left(\theta_1 A_2 + a/\theta_1\right)} \right] d\theta_1
\]

\[
\left[ \frac{1}{a_2} \right]^{[2(n-m)-\beta]/2} \text{Bessel K}[-2(n-m) + \beta, 2\sqrt{a_2 A_3}]
\]

\[
\int_{\beta_1}^{\beta_2} \frac{\alpha \beta}{\Gamma \beta} A_1 2^{\beta-1} \beta^n \ d\beta \ h^{-1}(T) \left[ \frac{1}{Q_3} \right]
\]

\[
= [K] \sum_{m=1}^{n-l} 2A_2^{-[2m-\beta]/2} \left[ \frac{1}{a_2} \right]^{[2(n-m)-\beta]/2} \text{Bessel K}[-2m + \beta - q_3, 2\sqrt{a_1 A_2}] 2A_3^{-[2(n-m)-\beta]/2}
\]

where \( \text{Bessel K}[-2(n-m) + \beta, 2\sqrt{a_2 A_3}] \) and \( \text{Bessel K}[-2(n-m) + \beta, 2\sqrt{a_1 A_2}] \) are same as in (18).

Minimizing expectation \( [E_{\theta_2} [L_5 (\theta_2, d)] \) and using posterior distribution (20), we get Bayes estimate of \( \theta_2 \) using General Entropy Loss Function as:

\[
\theta_{2E}^* = \left[ E(\theta_2^{-q_3}) \right]^{1/q_3}
\]

\[
= [K] \sum_{m=1}^{n-l} \int_{0}^{\infty} \left[ \theta_2^{2(n-m)-\beta-q_3-1} e^{-\left(\theta_2 A_3 + a/\theta_2\right)} \right] d\theta_2
\]

\[
2A_2^{-[2m-\beta]/2} \left[ \frac{1}{a_2} \right]^{[2(n-m)-\beta]/2} \text{Bessel K}[-2m + \beta, 2\sqrt{a_1 A_2}]
\]

\[
\int_{\beta_1}^{\beta_2} \frac{\alpha \beta}{\Gamma \beta} A_1 2^{\beta-1} \beta^n e^{-\beta a_1} d\beta \ h^{-1}(T) \left[ \frac{1}{Q_3} \right]
\]

\[
= [K] \sum_{m=1}^{n-l} 2A_3^{-[2(n-m)-\beta-q_3]/2} \left[ \frac{1}{a_2} \right]^{[2(n-m)-\beta-q_3]/2}
\]
Bessel $K[-2(n-m) + \beta - q_3, 2\sqrt{a_2\sqrt{A_3}}] 2A_2^{-[2m-\beta]/2}$

\[
\left[ \frac{j}{a_j} \right]^{-[2m-\beta]/2} Bessel \ K [-2m + \beta, 2\sqrt{a_1\sqrt{A_2}}]
\]

\[
\int_{\beta_1}^{\beta_2} \frac{a_{1}^{\beta} \ a_{2}^{\beta}}{\Gamma^2} \ A_1 \ 2^{\beta-1} \ \beta^n \ e^{-\beta q_3} \ d\beta \ h^{-1}(T) \ 
\]

\[
\frac{1}{q_3} \ (33)
\]

where

Bessel

$K[-2m + \beta, 2\sqrt{a_1\sqrt{A_2}}]$

and

Bessel $K[-2(n-m) + \beta - q_3, 2\sqrt{a_2\sqrt{A_3}}]$ are same as in (18).

Minimizing expectation $[E_{\beta} \ [L_5 (\beta, d)]]$ and using posterior distributions (21), we get Bayes estimate of $\theta_\beta$ using General Entropy Loss Function as:

\[
\beta_{E}^{*} = \left[ E(\beta^{-q_3}) \right]^{-\frac{1}{q_3}}
\]

\[
= \left[ K \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_{1}^{\beta} \ a_{2}^{\beta}}{\Gamma^2} \ A_1 \ 2^{\beta-1} \ \beta^n \ e^{-\beta q_3} \ d\beta \ h^{-1}(T) \right]^{-\frac{1}{q_3}} \ 
\]

\[
(34)
\]

5. CALCULATIONS FROM THE NUMERICAL STUDY:

We have generated 20 random observations from proposed Weighted Weibull Length Biased change point model. The first eight observations are with $\beta = 2.5$ and $\theta_1 = 0.005$ and next twelve are with $\beta = 2.5$ and $\theta_2 = 0.002$. Here, we note that $\theta_1$ and $\theta_2$ themselves were random observations from Inverted Gamma Distribution with prior means $\mu_1 = 0.05$, $\mu_2 = 0.02$ and variance $\sigma_1^2 = 0.000, \sigma_2^2 = 0.0008$ resulting in $a_1 = 0.0075$ and $a_2 = 0.0030$. These observations are given below in TABLE 1.
TABLE 1
GENERATED OBSERVATIONS FROM PROPOSED CHANGE POINT MODEL

<table>
<thead>
<tr>
<th>I</th>
<th>X_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.022</td>
</tr>
<tr>
<td>2</td>
<td>0.950</td>
</tr>
<tr>
<td>3</td>
<td>0.111</td>
</tr>
<tr>
<td>4</td>
<td>0.549</td>
</tr>
<tr>
<td>5</td>
<td>0.846</td>
</tr>
<tr>
<td>6</td>
<td>0.666</td>
</tr>
<tr>
<td>7</td>
<td>0.198</td>
</tr>
<tr>
<td>8</td>
<td>0.306</td>
</tr>
<tr>
<td>9</td>
<td>0.055</td>
</tr>
<tr>
<td>10</td>
<td>0.156</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I</th>
<th>X_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0.291</td>
</tr>
<tr>
<td>12</td>
<td>0.228</td>
</tr>
<tr>
<td>13</td>
<td>0.460</td>
</tr>
<tr>
<td>14</td>
<td>0.396</td>
</tr>
<tr>
<td>15</td>
<td>0.783</td>
</tr>
<tr>
<td>16</td>
<td>0.999</td>
</tr>
<tr>
<td>17</td>
<td>0.001</td>
</tr>
<tr>
<td>18</td>
<td>0.108</td>
</tr>
<tr>
<td>19</td>
<td>0.888</td>
</tr>
<tr>
<td>20</td>
<td>0.963</td>
</tr>
</tbody>
</table>

Now, we have calculated the values of posterior mean of \( m, \theta_1, \theta_2, \beta \). We have also calculated the posterior median and posterior mode of ‘m’. The results are shown below in TABLE 2.

TABLE 2

<table>
<thead>
<tr>
<th>Prior Density</th>
<th>Bayes Estimates of change point</th>
<th>Bayes Estimates of Posterior means of parameters ( \theta_1 ) and ( \theta_2 )</th>
<th>Bayes Estimates of Posterior means of parameters ( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Posterior Median</td>
<td>Posterior Mean</td>
<td>Posterior mode</td>
</tr>
<tr>
<td>Inverted Gamma Prior</td>
<td>8.13</td>
<td>8.33</td>
<td>8.13</td>
</tr>
</tbody>
</table>

We have also computed the Bayes Estimates \( m_i^*, m^*_e \) of \( m, \theta_{1L}^*, \theta_{1E}^*, \theta_{2L}^*, \theta_{2E}^* \) of \( \theta_1, \theta_2, \beta_L, \beta_E \) of \( \beta \).

Using the results for the data given in TABLE 3 and for different values of shape parameter \( q_1 \) and \( q_3 \), the results are shown below in TABLE 3 and TABLE 4.
TABLE 3

BAYES ESTIMATES USING LINEX LOSS FUNCTION

<table>
<thead>
<tr>
<th>Prior Density</th>
<th>(q_1)</th>
<th>(m_L^*)</th>
<th>(\theta_{1L}^*)</th>
<th>(\theta_{2L}^*)</th>
<th>(\beta_L^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverted Gamma Prior</td>
<td>0.09</td>
<td>8</td>
<td>0.05</td>
<td>0.023</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>8</td>
<td>0.05</td>
<td>0.022</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>0.20</td>
<td>8</td>
<td>0.05</td>
<td>0.021</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>7</td>
<td>0.03</td>
<td>0.018</td>
<td>2.2</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>6</td>
<td>0.02</td>
<td>0.014</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>9</td>
<td>0.09</td>
<td>0.027</td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td>-2.0</td>
<td>10</td>
<td>0.010</td>
<td>0.029</td>
<td>2.7</td>
</tr>
</tbody>
</table>

TABLE 4

BAYES ESTIMATES USING GENERAL ENTROPY LOSS FUNCTION

<table>
<thead>
<tr>
<th>Prior Density</th>
<th>(q_3)</th>
<th>(m_E^*)</th>
<th>(\theta_{1E}^*)</th>
<th>(\theta_{2E}^*)</th>
<th>(\beta_E^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverted Gamma prior</td>
<td>0.09</td>
<td>8</td>
<td>0.05</td>
<td>0.023</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>8</td>
<td>0.05</td>
<td>0.021</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>0.20</td>
<td>8</td>
<td>0.05</td>
<td>0.020</td>
<td>2.3</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>6</td>
<td>0.03</td>
<td>0.017</td>
<td>2.2</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>5</td>
<td>0.02</td>
<td>0.015</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>9</td>
<td>0.09</td>
<td>0.025</td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td>-2.0</td>
<td>10</td>
<td>0.10</td>
<td>0.028</td>
<td>2.8</td>
</tr>
</tbody>
</table>

Above table shows that for small values such as \(q_1 = 0.09, 0.10, 0.20\), Linex Loss Function is almost symmetric and nearly quadratic and the values of the Bayes Estimates under such a loss is not far from the posterior mean. TABLE 3 also shows that for \(q_1 = 1.2, 1.5\), Bayes Estimates are less than actual value of \(m = 8\).

For \(q_1 = q_3 = -1, -2\), we can clearly see that the Bayes Estimates are quite large than actual value \(m = 8\).

It can be seen from the TABLE 3 and TABLE 4 that the negative sign of shape parameter of loss functions reflects under estimation is more serious than that over the estimation. Thus, problem of under estimation can be solved by taking the value of shape parameters of Linex and General Entropy Loss Functions as negative.
TABLE 4 shows that for small values of $|q_3|$, $q_3 = 0.09, 0.10, 0.20$, the values of the Bayes Estimate obtained using General Entropy Loss Functions are not far from the posterior mean. TABLE 4 also shows that for $q_3 = 1.2, 1.5$, Bayes Estimates are less than actual value of $m = 8$.

Here, it is clearly seen from the TABLE 3 and TABLE 4 that positive sign of shape parameter of loss functions reflects over estimation is more serious than under estimation. Thus, problem of over estimation can be solved by taking the value of shape parameter of Linex and General Entropy Loss Functions as positive and high.

6. SENSITIVITY OF BAYES ESTIMATES

In this section, we have studied the sensitivity of the Bayes Estimates obtained with respect to change in the prior of the parameter. The mean values $\mu_1$ and $\mu_2$ and variances $\sigma_1^2$ and $\sigma_2^2$ have been used as prior information in computing the parameters of the prior. The results are shown below in TABLE 5.

TABLE 5
POSTERIOR MEAN $m^*$ FOR THE DATA GIVEN IN TABLE 2

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$m^*$</th>
<th>$m^*_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.005</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>0.005</td>
<td>0.006</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>0.005</td>
<td>0.008</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>0.007</td>
<td>0.002</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>0.002</td>
<td>0.002</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>0.4</td>
<td>0.002</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>0.002</td>
<td>0.004</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>0.003</td>
<td>0.005</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>0.004</td>
<td>0.006</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>
TABLE 5 leads to the conclusion that $m^*$ and $m^*_E$ are robust with respect to the correct choice of the prior density of $\theta_1(\theta_2)$ and a wrong choice of the prior density of $\theta_1(\theta_2)$. Moreover, they are also robust with respect to the change in the shape parameter of General Entropy Loss Function.

7. CONCLUSION:

The results in all the tables lead to the conclusion that performance of posterior means has better performance than that of $m^*_L$ and $m^*_E$ of change point. 60% values of posterior mean are closed to actual value of change point with correct choice of prior. 64% values of posterior median are closed to actual value of change point with correct choice of prior. 63% values of posterior mode are closed to actual value of change point with correct choice of prior. 67% values of $m^*_{L}$ are closed to actual value of change point with correct choice of prior. 68% values of $m^*_{E}$ are closed to actual value of change point with correct choice of prior.

8. REFERENCES:


