On GPR-Compactness in topological spaces

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Abstract: The purpose of this paper to introduce a new class of continuous functions like GPR-compactness functions and study some of their properties.

Key words: GPR-continuity, gpr-continuity, gpr-continuous, ect.

I. INTRODUCTION

Maki et al [5,6] defined $\alpha$-generalized closed sets and the notion of generalized preclosed set (briefly gp closed) sets in topological which are weaker forms of closed it is observed that every $\alpha g$-closed set is gp-[2] first defined a new closed set using regular-open set known as generalized pre regular closed. Palaniappan and Rao [8] defined the notion of regular generalized (briefly rg-closed) Gnanambal closed set (briefly gpr-closed). In this chapter we study the properties of gpr-closed sets and gp-continuous functions

II PRELIMINARIES :

Throughout this dissertation work $(X, \tau), (Y, \sigma)$ and $(Z,\eta)$ represent non-empty topological spaces on which no separation axioms are assumed unless explicitly stated, and they are simply written X, Y and Z respectively. For a subset A of $(X, \tau)$, the closure of A, the interior of A with respect to $\tau$ are denoted by $\text{cl}(A)$ and $\text{int}(A)$ respectively. The complement of A is denoted by $A^c$. The $\alpha$-closure of A is the smallest $\alpha$-closed set containing A and this is denoted by $\alpha cl(A)$.

Definition : A subset A of a topological space $(X, \tau)$ is called a

1. semi-open set [3] if $A \subseteq \text{cl(int}(A))$ and semi-closed set if $\text{int(cl}(A)) \subseteq A$.
2. pre-open set [4] if $A \subseteq \text{int(cl}(A))$ and pre-closed set if $\text{cl(int}(A)) \subseteq A$.
3. $\alpha$-open set [7] if $A \subseteq \text{int(cl(int}(A)))$ and $\alpha$-closed set if $\text{cl(int(cl}(A))) \subseteq A$
4. semi pre open set [1] if $A \subseteq \text{cl(int(cl}(A)))$ and semi pre closed [1] if $\text{int(cl(int}(A))) \subseteq A$
5. regular open set [9] if $A=\text{int(cl}(A))$ and a regular closed set if $A=\text{cl(int}(A))$. 
In this section we study the concept of GPR-compactness and GPR-connectedness using gpr-open sets and studied some of their characterizations.

**Definition 3.1:** A collection \( \{A_i : i \in I\} \) of gpr-open sets in a topological space \((X, \tau)\) is called a gpr-open cover of a subset \(A\) in \((X, \tau)\) if \( A \subseteq \bigcup_{i \in I} A_i \) holds.

**Definition 3.2:** A topological space \((X, \tau)\) is GPR-compact if every gpr-open cover of \((X, \tau)\) has a finite subcover.

**Definition 3.3:** A subset \(A\) of a topological space \((X, \tau)\) is said to be GPR-compact relative to \((X, \tau)\), if for every collection \(\{A_i : i \in I\}\) of gpr-open subsets of gpr-open subsets of \((X, \tau)\) such that \( A \subseteq \bigcup_{i \in I} A_i \) there exists a finite subset \(I_0\) of \(I\) such that \( A \subseteq \bigcup_{i \in I_0} A_i \).

**Definition 3.4:** A subset \(A\) of a topological space \((X, \tau)\) is called GPR-compact if \(A\) is GPR-compact of the subspace of \((X, \tau)\).

**Theorem 3.5:** A gpr-closed subset of a GPR-compact space \((X, \tau)\) is GPR-compact relative to \((X, \tau)\).

**Proof:** Let \(A\) be gpr-closed subset of a GPR-compact space \(X\). Then \(X - A\) is gpr-open. Let \(\Omega\) be a gpr-open cover for \(A\). Then \(\{\Omega, X - A\}\) is a gpr-open cover for \(X\). Since \(X\) is GPR-compact, it has a finite subcover, say, \(\{p_1, p_2, ..., p_n\} = \Omega_i\). If \(X - A \notin \Omega\), then \(\Omega_i\) is a finite subcover of \(A\). If \(X - A \in \Omega_i\), then \(\Omega_i\) is a subcover of \(A\). Thus \(A\) is a GPR-compact relative to \((X, \tau)\).

**Theorem 3.6:** The image of a GPR-compact space under GPR-continuous map is compact.

**Proof:** Let \(f: X \to Y\) be gpr-continuous map from a GPR-compact space \((X, \tau)\) onto a topological space \((Y, \mu)\). Let \(\{A_i : i \in I\}\) be an open cover of \((Y, \sigma)\). Since \(f\) is gpr-continuous, \(\{f^{-1}(A_i) : i \in I\}\) is an gpr-open cover of \((X, \tau)\). Since \((X, \tau)\) is GPR-compact, the \(\omega\alpha\)-open cover of \((X, \tau)\), \(\{f^{-1}(A_i) : i \in I\}\) has a finite subcover say \(\{f^{-1}(A_i) : i = 1, ..., n\}\). Therefore \(X = \bigcup_{i=1}^{n} f^{-1}(A_i)\) which implies \(f(X) = \bigcup_{i=1}^{n} A_i\), this implies \(Y = \bigcup_{i=1}^{n} A_i\) that is, \(\{A_1, A_2, ..., A_n\}\) is a finite subcover of \(\{A_i : i \in I\}\) for \((Y, \mu)\). Hence \((Y, \mu)\) is compact.

**Theorem 3.7:** If a map \(f: X \to Y\) is gpr-irresolute and a subset \(S\) of \(X\) is GPR-compact relative to \((X, \tau)\), then the image \(f(S)\) is GPR-compact relative to \((Y, \sigma)\).

**Proof:** Let \(\{A_i : i \in I\}\) be a collection of gpr-open sets in \((Y, \sigma)\) such that \(f(S) \subseteq \bigcup_{i \in I} A_i\). Then \(S \subseteq \bigcup_{i=1}^{n} f^{-1}(A_i)\), where \(\{f^{-1}(A_i) : i \in I\}\) is gpr-open set in \((X, \tau)\). Since \(S\) is GPR-compact relative to \((X, \tau)\), there exist finite subcollections \(\{A_1, A_2, ..., A_n\}\) such that \(S \subseteq \bigcup_{i=1}^{n} f^{-1}(A_i)\). That is \(f(S) \subseteq \bigcup_{i=1}^{n} A_i\). Hence \(f(S)\) is GPR-compact relative to \((Y, \sigma)\).
Definition 3.8: A topological space \((X, \tau)\) is said to be GPR-connected if \((X, \tau)\) cannot be written as a disjoint union of two non-empty gpr-open sets.

A subset of \((X, \tau)\) is GPR-connected if it is GPR-connected as a subspace.

Theorem 3.9: For a topological space \((X, \tau)\) the following are equivalent

(i) \((X, \tau)\) is GPR-connected.

(ii) The only subsets of \((X, \tau)\) which are both gpr-open and gpr-closed are the empty set \(\phi\) and \(X\).

(iii) Each gpr-continuous map of \((X, \tau)\) into a discrete space \((Y, \mu)\) with at least two points is a constant map.

Proof: (i) \(\Rightarrow\) (ii) Let \(G\) be a gpr-open and a gpr-closed subset of \((X, \tau)\). Then \(X = G \cup (X - G)\), a disjoint union of two non-empty gpr-open sets which contradicts to the fact that \((X, \tau)\) is GPR-connected. Hence \(G = \phi\) or \(X\).

(ii) \(\Rightarrow\) (i) Suppose that \(X = A \cup B\) where \(A\) and \(B\) are disjoint non-empty gpr-open subsets of \((X, \tau)\). Since \(A = X - B\) then \(A\) is both gpr-closed and gpr-open. By assumption \(A = \phi\) or \(X\), which is a contradiction. Hence \((X, \tau)\) is GPR-connected.

(ii) \(\Rightarrow\) (iii) Let \(f: (X, \tau) \to (Y, \mu)\) be a gpr-continuous map, where \((Y, \mu)\) is discrete space with at least two points. Then \(f^{-1}(\{y\})\) is gpr-closed and gpr-open for each \(y \in Y\). That is \((X, \tau)\) is covered by gpr-closed and gpr-open covering \(\{f^{-1}(\{y\}) : y \in Y\}\). By assumption, \(f^{-1}(\{y\}) = \phi\) or \(X\) for each \(y \in Y\). If \(f^{-1}(\{y\}) = \phi\) for each \(y \in Y\), then \(f\) fails to be a map. Therefore there exist at least one point say \(f^{-1}(\{y_1\}) \neq \phi\), \(y_1 \in Y\) such that \(f^{-1}(\{y_1\}) = X\). This shows that \(f\) is a constant map.

(iii) \(\Rightarrow\) (ii) Let \(G\) be both gpr-closed and gpr-open in \((X, \tau)\). Suppose \(G \neq \phi\). Let \(f: (X, \tau) \to (Y, \mu)\) be a gpr-continuous map defined by \(f(G) = \{a\}\) and \(f(X - G) = \{b\}\) where \(a \neq b\) and \(a, b \in Y\). By assumption, \(f\) is constant and so \(G = X\).

Theorem 3.10: Every GPR-connected space is connected.

Proof: Let \((X, \tau)\) be an GPR-connected space. Suppose that \((X, \tau)\) is not connected. Then \(X = A \cup B\) where \(A\) and \(B\) are disjoint non-empty open subsets of \((X, \tau)\). Then \(A\) and \(B\) are gpr-open disjoint sets and \(X = A \cup B\) where \(A\) and \(B\) are disjoint non-empty gpr-open subsets of \((X, \tau)\). This contradicts to the fact that \((X, \tau)\) is GPR-connected and so \((X, \tau)\) is connected.

The converse of the above theorem need not be true as seen from the following example.

Example 3.11: Let \(X = \{a, b, c\}\) and \(\tau = \{\phi, \{a\}, \{a, b\}, X\}\). Then \((X, \tau)\) is not GPR-connected space but it is connected space because every subset of \(X\) is gpr-open. The only clopen sets of \(X\) are \(\phi, X\). Therefore \(X\) is connected.

Theorem 3.12: If \(f: (X, \tau) \to (Y, \sigma)\) is gpr-continuous surjection and \((X, \tau)\) is GPR-connected, then \((Y, \sigma)\) is connected.
**Proof:** Suppose that \((Y, \sigma)\) is not connected. Let \(Y = A \cup B\) where \(A\) and \(B\) are disjoint non-empty open subsets in \((Y, \sigma)\). Since \(f\) is gpr-continuous, \(X = f^{-1}(A) \cup f^{-1}(B)\) where \(f^{-1}(A)\) and \(f^{-1}(B)\) are disjoint non-empty gpr-open subsets in \((X, \tau)\). This contradicts to the fact that \((X, \tau)\) is GPR-connected. Hence \((Y, \sigma)\) is connected.

**Theorem 3.13:** If \(f: (X, \tau) \to (Y, \sigma)\) is gpr-irresolute surjection and \((X, \tau)\) is GPR-connected, then \((Y, \sigma)\) is GPR-connected.

**Proof:** Suppose that \((Y, \sigma)\) is not GPR-connected. Let \(Y = A \cup B\) where \(A\) and \(B\) are disjoint non-empty gpr-open subsets in \((Y, \sigma)\). Since \(f\) is gpr-continuous and surjection, \(X = f^{-1}(A) \cup f^{-1}(B)\) where \(f^{-1}(A)\) and \(f^{-1}(B)\) are disjoint non-empty gpr-open subsets in \((X, \tau)\). This contradicts to the fact that \((X, \tau)\) is GPR-connected. Hence \((Y, \sigma)\) is GPR-connected.

### IV REFERENCES


