ON \( \tau_1 \tau_2#RG\)-CONTINUOUS IN BITOPOLIGICAL SPACES AND \( \tau_1 \tau_2#RG\)-IRRESOLUTE FUNCTIONS

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Abstract

In this paper we introduce \( \tau_1 \tau_2#rg\)-closed sets and \( \tau_1 \tau_2#rg\)-open sets in bitopological spaces and established their relationships with some generalized sets in bitopological spaces. The aim of this paper is to introduce \( \tau_1 \tau_2#rg\)-continuous functions and \( \tau_1 \tau_2#rg\)-irresolute functions by using \( \tau_1 \tau_2#rg\)-closed sets and characterize their basic properties.

Keywords: \( \tau_1 \tau_2#rg\)-closed; \( \tau_1 \tau_2#rg\)-open; \( \tau_1 \tau_2#rg\)-continuous; \( \tau_1 \tau_2#rg\)-irresolute.

1. Introduction

The concept of continuity is connected with the concept of topology. A weaker form of continuous functions called generalized continuous (briefly, \( g\)-continuous) maps was introduced and studied by Balachandran [1]. Then many researchers studied on generalizations of continuous maps. Recently, Sivanthi and Thilaga Leevathi [2] introduced and studied the properties of \( \tau_1 \tau_2#rg\)-closed sets. The purpose of this paper is to introduce the concept of \( \tau_1 \tau_2#rg\)-continuous and \( \#rg\)-irresoluteness that are characterized and their relationship with weak and generalized continuity are investigated.

2. Preliminaries

Throughout this paper \((X; \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) (or briefly, \(X\) and \(Y\)) represents a bitopological space on which no separation axiom is assumed unless otherwise mentioned. For a subset \(A\) of a bitopological space \(X\), \(\tau_2\text{cl}(A)\) and \(\tau_1\text{int}(A)\) denote the \(\tau_2\) closure of \(A\) and the \(\tau_1\) interior of \(A\), respectively. \(X \setminus A\) or \(A^c\) denotes the complement of \(A\) in \(X\). We recall the following definitions and results.

**Definition 2.1** A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called:

1. \(\tau_1\tau_2\) preopen set if \(A \subseteq \tau_1\text{int}\tau_2\text{cl}(A)\) and a \(\tau_1\tau_2\) preclosed set if \(\tau_2\text{cl}\tau_1\text{int}(A) \subseteq A\).
2. \(\tau_1\tau_2\) semiopen set if \(A \subseteq \tau_2\text{cl}\tau_1\text{int}(A)\) and a \(\tau_1\tau_2\) semiclosed set if \(\tau_1\text{int}\tau_2\text{cl}(A) \subseteq A\).
3. \(\tau_1\tau_2\) regular open set if \(A = \tau_1\text{int}\tau_2\text{cl}(A)\) and a \(\tau_1\tau_2\) regular closed set if \(A = \tau_2\text{cl}\tau_1\text{int}(A)\).
4. \(\tau_1\tau_2\) \(\tau_2\)-open set if \(A\) is a finite union of regular open sets.
5. \(\tau_1\tau_2\) regular semi open if there is a \(\tau_1\) regular open \(U\) such \(U \subseteq A \subseteq \tau_2\text{cl}(U)\).

**Definition 2.2** A subset \(A\) of \((X, \tau_1, \tau_2)\) is called

1. \(\tau_1\tau_2\) generalized closed set (briefly, \(\tau_1\tau_2\)-g-closed) if \(\tau_2\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
2. \(\tau_1\tau_2\) regular generalized closed set (briefly, \(\tau_1\tau_2\)-rg-closed) if \(\tau_2\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_1\)-regular open in \(X\).
3. \(\tau_1\tau_2\) generalized preregular closed set (briefly, \(\tau_1\tau_2\)-gpr-closed) if \(\tau_2\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_1\)-regular open in \(X\).
4. \(\tau_1\tau_2\) regular weakly generalized closed set (briefly, \(\tau_1\tau_2\)-wg-closed) if \(\tau_2\text{cl}\tau_1\text{int}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_1\)-regular open in \(X\).
(5) \( \tau_1 \cap \tau_2 \) is rw-closed if \( \tau_2 \) cl(\( A \)) \( \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \) regular semi open.
(6) \( \tau_1 \cap \tau_2 \) is \( rg \)-closed if \( \tau_2 \) cl(\( A \)) \( \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \) rw-open.

The complements of the above mentioned closed sets are their respective open sets.

**Definition 2.3** A map \( f : X \rightarrow Y \) is called \( \tau_1 \cap \tau_2 \) g-continuous [1] (resp. \( \tau_1 \cap \tau_2 \) \( rg \)-continuous ) if \( f^{-1}(V) \) is \( g \)-closed (resp. \( \tau_1 \cap \tau_2 \) \( rg \)-closed) in \( X \) for every closed subset \( V \) of \( Y \).

**Definition 2.4** For a subset \( A \) of a space \((X, \tau_1, \tau_2)\), \( \tau_1 \cap \tau_2 \) \( #rg \) \( \cap \) cl(A) = \( \cap \{ F : A \subseteq F ; F \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-closed in \( X \} \) is called the \( \tau_1 \cap \tau_2 \) \( #rg \)-closure of \( A \).

**Definition 2.5** Let \((X; \tau_1, \tau_2)\) be a bitopological space and \( \tau_1 \cap \tau_2 \) \( #rg \) \( \cap \) cl(A) \( = \{ V \subseteq X : \tau_1 \cap \tau_2 \) \( #rg \) \( \cap \) cl(X \( \setminus \) V) \( = \) X \( \setminus \) V \).

**Lemma 2.6** For any \( x \in X \), \( \tau_1 \cap \tau_2 \) \( #rg \)-closed if and only if \( V \cap A \neq \emptyset \) for every \( \tau_1 \cap \tau_2 \) \( #rg \)-open set \( V \) containing \( x \).

**Lemma 2.7** Let \( A \) and \( B \) be subsets of \((X; \tau_1, \tau_2)\). Then:
(1) \( \#rg \cap \tau_2 \) cl(\( \emptyset \)) = \( \emptyset \) and \( \tau_1 \cap \tau_2 \) \( #rg \) \( \cap \) - \( \tau_2 \) cl(X) = X.
(2) If \( A \subseteq B \), then \( \tau_1 \cap \tau_2 \) \( #rg \) \( \cap \) \( \tau_2 \) cl(\( A \)) \( \subseteq \) \( \tau_1 \cap \tau_2 \) \( #rg \) \( \cap \) \( \tau_2 \) cl(\( B \)).
(3) \( A \subseteq \tau_1 \cap \tau_2 \) \( #rg \) \( \cap \) \( \tau_2 \) cl(\( A \)).
(4) If \( A \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-closed, then \( \tau_1 \cap \tau_2 \) \( #rg \) \( \cap \) cl(A) = A.
(5) \( \tau_1 \cap \tau_2 \) \( #rg \)-closure of a set \( A \) is not always \( \tau_1 \cap \tau_2 \) \( #rg \)-closed.

**Remark 2.8** Suppose \( \tau_1 \cap \tau_2 \) \( #rg \) is a bitopology. If \( A \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-closed in \((X; \tau_1, \tau_2)\), then \( A \) is closed in \((X, \tau_1 \cap \tau_2 \) \( #rg \)).

**Lemma 2.9** A set \( A \subseteq X \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-open if and only if \( F \subseteq \tau_1 \) \( int \) \( A \) whenever \( F \subseteq A \), \( F \) is \( \tau_1 \cap \tau_2 \) \( rw \)-closed.

3. \( \tau_1 \cap \tau_2 \) \( #RG \)-Continuous Functions

In this section, we introduce and study \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous functions.

**Definition 3.1** A function \( f : X \rightarrow Y \) is called \( #rg \)-continuous if \( f^{-1}(V) \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-closed in \((X, \tau_1, \tau_2)\) for every closed subset \( V \) of \((Y, \sigma_1, \sigma_2)\).

**Theorem 3.2** Every continuous map is \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous map.

**Proof** Let \( f : X \rightarrow Y \) is continuous map then for every closed set \( A \) in \( Y \), \( f^{-1}(A) \) is closed in \( X \). Since every closed set is \( \tau_1 \cap \tau_2 \) \( #rg \)-closed, \( f^{-1}(A) \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-closed in \( X \). Hence \( f \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous map.

**Example 3.3** Let \( X = \{ a, b, c \} \) with topologies \( \tau_1 = \{ \emptyset, \{ a \}, \{ a, b \}, X \} \) and \( \tau_2 = \{ \emptyset, \{ a \}, \{ a, b \}, X \} \). Let \( Y = \{ 1, 2, 3, 4, 5 \} \) with topologies \( \sigma_1 = \{ \emptyset, \{ 1 \}, \{ 2, 3, 4, 5 \}, Y \} \) and \( \sigma_2 = \{ \emptyset, \{ 2 \}, \{ 3 \}, \{ 2, 3 \}, \{ 2, 3, 4, 5 \}, \emptyset \} \). A function \( F : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is defined as follows: \( F(a) = \{ 2, 3 \}, F(b) = \{ 1, 2 \}, F(c) = \{ 1, 4, 5 \} \). Then, \( F \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous.

**Corollary 3.1** Every \( \tau_1 \cap \tau_2 \) \( #rg \)-regular continuous map is \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous.

**Proof** Follows from Theorem 3.2 and the fact that every \( \tau_1 \cap \tau_2 \) \( #rg \)-regular continuous map is \( \tau_1 \cap \tau_2 \) continuous.

**Theorem 3.4** Every \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous map is \( \tau_1 \cap \tau_2 \) \( g \)-continuous map (resp. \( \tau_1 \cap \tau_2 \) \( rg \)-continuous).

**Proof** Suppose \( f : X \rightarrow Y \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous. Let \( V \) be a closed set in \( Y \). Since \( f \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous, then \( f^{-1}(V) \) is \( \tau_1 \cap \tau_2 \) \( #rg \)-closed set in \( X \). Since every \( \tau_1 \cap \tau_2 \) \( #rg \)-closed set is \( \tau_1 \cap \tau_2 \) \( g \)-closed (resp. \( \tau_1 \cap \tau_2 \) \( rg \)-closed) set, then \( f^{-1}(V) \) is also \( \tau_1 \cap \tau_2 \) \( g \)-closed (resp. \( \tau_1 \cap \tau_2 \) \( rg \)-closed) set in \( X \). Thus \( f \) is \( \tau_1 \cap \tau_2 \) \( g \)-continuous (resp. \( \tau_1 \cap \tau_2 \) \( rg \)-continuous).

The converse of the above theorem is not necessarily true as seen from the following example.

**Example 3.5** Let \( X = Y = \{ a, b, c \} \) with topologies \( \tau_1 = \{ \emptyset, \{ a \}, X \}, \tau_2 = \{ \emptyset, \{ a \}, \{ a, b \}, X \}, \sigma_1 = \{ \emptyset, \{ b, c \}, \{ a, c \}, \emptyset \} \) and \( \sigma_2 = \{ \emptyset, \{ b, c \}, \{ a, c \}, \emptyset \} \). Define \( f : X \rightarrow Y \) by \( f(a) = b, f(b) = a \) and \( f(c) = c \) then \( f \) is \( \tau_1 \cap \tau_2 \) \( g \)-continuous but not \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous.

**Corollary 3.2** Every \( \tau_1 \cap \tau_2 \) \( #rg \)-continuous is \( \tau_1 \cap \tau_2 \) \( rwg \)-continuous and \( \tau_1 \cap \tau_2 \) \( gpr \)-continuous.
Proof Follows from Theorem 3.4 and the fact that every $\tau_1 \tau_2 \#_{rg}$-continuous map is $\tau_1 \tau_2 \#_{rg}$-continuous and $\tau_1 \tau_2 \#_{gpr}$-continuous.

Corollary 3.3 Every $\tau_1 \tau_2 \#_{rg}$-continuous is $\tau_1 \tau_2 \#_{gs}$-continuous.

Proof Follows from Theorem 3.4 and the fact that every $g$-continuous map is $gs$-continuous.

Corollary 3.4 Every $\tau_1 \tau_2 \#_{rg}$-continuous is $\tau_1 \tau_2 \#_{gs}$-continuous.

Proof Follows from Corollary 3.3 and the fact that every $\tau_1 \tau_2 \#_{gs}$-continuous map is $\tau_1 \tau_2 \#_{gs}$-continuous.

Theorem 3.6 Let $f : X \to Y$ be a function. Then the following are equivalent:
(1) $f$ is $\tau_1 \tau_2 \#_{rg}$-continuous,
(2) The inverse image of each open set in $Y$ is $\tau_1 \tau_2 \#_{rg}$-open in $X$.
(3) The inverse image of each closed set in $Y$ is $\tau_1 \tau_2 \#_{rg}$-closed in $X$.

Proof
Suppose (1) holds. Let $G$ be open in $Y$. Then $Y \setminus G$ is closed in $Y$. By (1) $f^{-1}(Y \setminus G)$ is $\tau_1 \tau_2 \#_{rg}$-closed in $X$. But $f^{-1}(Y \setminus G) = X \setminus f^{-1}(G)$ which is $\tau_1 \tau_2 \#_{rg}$-closed in $X$. Therefore $f^{-1}(G)$ is $\tau_1 \tau_2 \#_{rg}$-open in $X$. This proves (1) $\Rightarrow$ (2).

Suppose (2) holds. Let $V$ be any closed set in $Y$. Then $Y \setminus V$ is open in $Y$. By (2), $f^{-1}(Y \setminus V)$ is $\tau_1 \tau_2 \#_{rg}$-open. But $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ which is $\tau_1 \tau_2 \#_{rg}$-open. Therefore $f^{-1}(V)$ is $\tau_1 \tau_2 \#_{rg}$-closed. This proves (2) $\Rightarrow$ (3).

The implication (3) $\Rightarrow$ (1) follows from Definition 3.1.

Theorem 3.7 If a function $f : X \to Y$ is $\tau_1 \tau_2 \#_{rg}$-continuous, then $f(\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(A)) \subseteq \tau_2 \text{ cl}(f(A))$ for every subset $A$ of $X$.

Proof
Let $f : X \to Y$ be $\tau_1 \tau_2 \#_{rg}$-continuous. Let $A \subseteq X$. Then $\tau_2 \text{ cl}(f(A))$ is closed in $Y$. Since $f$ is $\tau_1 \tau_2 \#_{rg}$-continuous, $f^{-1}(\tau_2 \text{ cl}(f(A)))$ is $\tau_1 \tau_2 \#_{rg}$-closed in $X$ and $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\tau_2 \text{ cl}(f(A)))$, implies $\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(A) \subseteq f^{-1}(\tau_2 \text{ cl}(f(A)))$. Hence $f(\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(A)) \subseteq \tau_2 \text{ cl}(f(A))$.

Theorem 3.8 Let $X$ be a space in which every singleton set is $\tau_1 \tau_2 \text{rw}$-closed. Then $f : X \to Y$ is $\tau_1 \tau_2 \#_{rg}$-continuous, if $x \in \tau_1 \text{ int}(f^{-1}(V))$ for every open subset $V$ of $Y$ contains $f(x)$.

Proof
Suppose $f : X \to Y$ is $\tau_1 \tau_2 \#_{rg}$-continuous. Fix $x \in X$ and an open set $V$ in $X$ such that $f(x) \in V$. Then $f^{-1}(V)$ is $\tau_1 \tau_2 \#_{rg}$-open. Since $x \in (f^{-1}(V))$ and $\{x\}$ is $\tau_1 \tau_2 \text{rw}$-closed, $x \in \tau_1 \text{ int}(f^{-1}(V))$ by Lemma 2.9.

Conversely, assume that $x \in \tau_1 \text{ int}(f^{-1}(V))$ for every open subset $V$ of $Y$ containing $f(x)$. Let $V$ be an open set in $Y$. Suppose $F \subseteq f^{-1}(V)$ and $F$ is $\tau_1 \tau_2 \text{rw}$-closed. Let $x \in F$, then $\forall x \in V$ so that $x \in \tau_1 \text{ int}(f^{-1}(V))$. That implies $F \subseteq x \in \tau_1 \text{ int}(f^{-1}(V))$. Therefore by Lemma 2.9, $f^{-1}(V)$ is $\tau_1 \tau_2 \#_{rg}$-open. This proves that $f$ is $\#_{rg}$-continuous.

Theorem 3.9 Let $f : X \to Y$ be a function. Let $X$ and $Y$ be any two spaces such that $\tau_1 \tau_2 \#_{rg}$ is a bitopology on $X$. Then the following statements are equivalent:
(1) For every subset $A$ of $X$, $f(\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(A)) \subseteq \tau_2 \text{ cl}(f(A))$ holds,
(2) $f : (X; \tau_1 \tau_2 \#_{rg}) \to (Y, \sigma_1, \sigma_2)$ is continuous.

Proof
Suppose (1) holds. Let $A$ be closed in $Y$. By hypothesis $f(\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(f^{-1}(A))) \subseteq \tau_2 \text{ cl}(f((f^{-1}(A))) \subseteq \tau_2 \text{ cl}(A) = A$. i.e., $\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(f^{-1}(A)) \subseteq f^{-1}(A)$. Also $f^{-1}(A) \subseteq \tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(f^{-1}(A))$. Hence, $\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(f^{-1}(A)) = f^{-1}(A)$. This implies $(f^{-1}(A)) \in \tau_1 \tau_2 \#_{rg}$. Thus $f^{-1}(A)$ is closed in $(X; \tau_1 \tau_2 \#_{rg})$ and so $f$ is continuous. This proves (2).

Suppose (2) holds. For every subset $A$ of $X$, $\tau_2 \text{ cl}(f(A))$ is closed in $Y$. Since $f : (X, \tau_1 \tau_2 \#_{rg}) \to (Y, \sigma_1, \sigma_2)$ is continuous, $(f^{-1}(\tau_2 \text{ cl}(f(A))))$ is closed in $(X, \tau_1 \tau_2 \#_{rg})$ that implies by Definition 2.5 $\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(f^{-1}(\tau_2 \text{ cl}(f(A)))) = (f^{-1}(\tau_2 \text{ cl}(f(A))))$. Now we have, $A \subseteq (f^{-1}(\tau_2 \text{ cl}(f(A))) \subseteq f^{-1}(\tau_2 \text{ cl}(f(A)))$ and by Lemma 2.7 (2), $\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(A) \subseteq \tau_1 \tau_2 \#_{rg} \subseteq \tau_2 \text{ cl}(f(A)) = (f^{-1}(\tau_2 \text{ cl}(f(A))))$. Therefore $f(\tau_1 \tau_2 \#_{rg} - \tau_2 \text{ cl}(A)) \subseteq \tau_2 \text{ cl}(f(A))$.

Theorem 3.10 Let $X$, $Y$ and $Z$ be bitopological spaces such that $\sigma_{\tau_1 \tau_2 \#_{rg}} = \sigma$. Let $f : X \to Y$ and $g : Y \to Z$ be $\tau_1 \tau_2 \#_{rg}$-continuous functions. Then the composition $g \circ f : X \to Z$ is $\tau_1 \tau_2 \#_{rg}$-continuous.
Proof Let $V$ be closed in $(Z, \mu_1, \mu_2)$. Since $g$ is $\tau_1 \tau_2$ #rg-continuous, $g^{-1}(V)$ is $\tau_1 \tau_2$ #rg-closed in $Y$. Since $\sigma_{\tau_1 \tau_2}$ #rg $= \sigma$, by Remark 2.8, $g^{-1}(V)$ is closed in $Y$. Since $f$ is $\tau_1 \tau_2$ #rg-continuous, $(f^{-1}(g^{-1}(V)))$ is $\tau_1 \tau_2$ #rg-closed.
i.e. $(g \circ f)^{-1}(V)$ is $\tau_1 \tau_2$ #rg-closed in $X$. Therefore $g \circ f$ is $\tau_1 \tau_2$ #rg-continuous.

4. $\tau_1 \tau_2$ #RG-Irresolute Functions

In this section $\tau_1 \tau_2$ #rg-irresolute function is introduced and their basic properties are discussed.

Definition 4.1 A function $f : X \rightarrow Y$ is called $\tau_1 \tau_2$ #rg-irresolute if $f^{-1}(V)$ is $\tau_1 \tau_2$ #rg-closed in $X$ for every $\tau_1 \tau_2$ #rg-closed subset $V$ of $Y$.

Theorem 4.2 Every $\tau_1 \tau_2$ #rg-irresolute function is $\tau_1 \tau_2$ #rg-continuous but converse is not necessarily true.

Proof Suppose $f : X \rightarrow Y$ is $\tau_1 \tau_2$ #rg-irresolute. Let $V$ be any closed subset of $Y$, then $V$ is $\tau_1 \tau_2$ #rg-closed set in $Y$. Since $f$ is $\tau_1 \tau_2$ #rg-irresolute, $f^{-1}(V)$ is $\tau_1 \tau_2$ #rg-closed in $X$. Hence $f$ is $\tau_1 \tau_2$ #rg-continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 4.3 Let $X = Y = \{a, b, c, d\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$. Define $f : X \rightarrow Y$ by identity mapping then $f$ is $\tau_1 \tau_2$ #rg-continuous but not $\tau_1 \tau_2$ #rg-irresolute.

Theorem 4.4 If map $f : X \rightarrow Y$ is $\tau_1 \tau_2$ #rg-continuous map and $Y$ is $T_{\tau_1 \tau_2}$ #rg-space, then $f$ is $\tau_1 \tau_2$ #rg-irresolute.

Proof Let $f : X \rightarrow Y$ is $\tau_1 \tau_2$ #rg-continuous map then inverse image of every closed set in $Y$ is $\tau_1 \tau_2$ #rg-closed set in $X$. Since $Y$ is $T_{\#rg}$-space, inverse image of every $\tau_1 \tau_2$ #rg-closed set in $Y$ is $\tau_1 \tau_2$ #rg-closed set in $X$. i.e., $f$ is $\tau_1 \tau_2$ #rg-irresolute.

Theorem 4.5 Let $f : X \rightarrow Y$ be $\tau_1 \tau_2$ #rg-open and closed. Then $f$ maps a $\tau_1 \tau_2$ #rg-closed set in $X$ into a $\tau_1 \tau_2$ #rg-closed set in $Y$.

Proof Let $A$ be $\tau_1 \tau_2$ #rg-closed in $X$. Let $f(A) \subseteq U$, where $U$ is $\tau_1 \tau_2$ #rg-open. Then $A \subseteq f^{-1}(U)$. Since $f$ is $\tau_1 \tau_2$ #rg-irresolute, $f^{-1}(U)$ is $\tau_1 \tau_2$ #rg-open in $X$. Since $A$ is $\tau_1 \tau_2$ #rg-closed, $\tau_2 cl(A) \subseteq f^{-1}(U)$ that implies $f(\tau_2 cl(A)) \subseteq \emptyset$.

Since $f$ is closed $f(\tau_2 cl(A))$ is closed that implies $\tau_2 cl(f(A)) \subseteq \tau_2 cl(f(\tau_2 cl(A))) = f(\tau_2 cl(A)) \subseteq U$. Hence $f(A)$ is $\tau_1 \tau_2$ #rg-closed in $Y$.

Theorem 4.6 Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be any two functions. Let $h = g \circ f$. Then:

1. $h$ is $\tau_1 \tau_2$ #rg-continuous if $f$ is $\tau_1 \tau_2$ #rg-irresolute and $g$ is $\tau_1 \tau_2$ #rg-continuous.
2. $h$ is $\tau_1 \tau_2$ #rg-irresolute if both $f$ and $g$ are both $\tau_1 \tau_2$ #rg-irresolute and,
3. $h$ is $\tau_1 \tau_2$ #rg-continuous if $g$ is continuous and $f$ is $\tau_1 \tau_2$ #rg-continuous.

Proof Let $V$ be closed in $Z$. (1) Suppose $f$ is $\tau_1 \tau_2$ #rg-irresolute and $g$ is $\tau_1 \tau_2$ #rg-continuous. Since $g$ is $\tau_1 \tau_2$ #rg-continuous, $g^{-1}(V)$ is $\tau_1 \tau_2$ #rg-closed in $Y$. Since $f$ is $\tau_1 \tau_2$ #rg-irresolute, using Definition 4.1, $f^{-1}(g^{-1}(V))$ is $\tau_1 \tau_2$ #rg-closed in $X$. This proves (1).

(2) Let $f$ and $g$ be both $\tau_1 \tau_2$ #rg-irresolute. Then $g^{-1}(V)$ is $\tau_1 \tau_2$ #rg-closed in $Y$. Since $f$ is $\tau_1 \tau_2$ #rg-irresolute, using Definition 4.1 $f^{-1}(g^{-1}(V))$ is #rg-closed in $X$. This proves (2).

(3) Let $g$ be continuous and $f$ be $\tau_1 \tau_2$ #rg-continuous. Then $g^{-1}(V)$ is closed in $Y$. Since $f$ is $\tau_1 \tau_2$ #rg-continuous, using Definition 3.1, $f^{-1}(g^{-1}(V))$ is $\tau_1 \tau_2$ #rg-closed in $X$. This proves (3).

The next theorem follows easily as a direct consequence of definitions.

Theorem 4.7 A function $f : X \rightarrow Y$ is $\tau_1 \tau_2$ #rg-irresolute if and only if the inverse image of every $\tau_1 \tau_2$ #rg-open set in $Y$ is $\tau_1 \tau_2$ #rg-open in $X$.

Definition 4.8 A function $f : X \rightarrow Y$ is said to be $\tau_1 \tau_2$ #rg-closed (resp. $\tau_1 \tau_2$ #rg-open) if for every #rg-closed (resp. $\tau_1 \tau_2$ #rg-open) set $U$ of $X$ the set $f(U)$ is $\tau_1 \tau_2$ #rg-closed (resp. $\tau_1 \tau_2$ #rg-open) in $Y$. 

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Theorem 4.9 Let \( f : X \to Y \) be a bijection. Then the following are equivalent:

1. \( f_1 \) is \( \tau_1 \tau_2 \#rg\)-open,
2. \( f_1 \) is \( \tau_1 \tau_2 \#rg\)-closed,
3. \( f_1^{-1} \) is \( \tau_1 \tau_2 \#rg\)-irresolute.

**Proof** Suppose \( f_1 \) is \( \tau_1 \tau_2 \#rg\)-open. Let \( F \) be \( \tau_1 \tau_2 \#rg\)-closed in \( X \). Then \( X \setminus F \) is \( \tau_1 \tau_2 \#rg\)-open. By Definition 4.8, \( f(X \setminus F) \) is \( \tau_1 \tau_2 \#rg\)-open. Since \( f \) is a bijection, \( Y \setminus f(F) \) is \( \tau_1 \tau_2 \#rg\)-open in \( Y \). Therefore \( f_1 \) is \( \tau_1 \tau_2 \#rg\)-closed.

This proves (1) \( \Rightarrow \) (2).

Let \( g = f_1^{-1} \). Suppose \( f_1 \) is \( \tau_1 \tau_2 \#rg\)-closed. Let \( V \) be \( \tau_1 \tau_2 \#rg\)-open in \( X \). Then \( X \setminus V \) is \( \tau_1 \tau_2 \#rg\)-closed in \( X \).

Since \( f_1 \) is \( \tau_1 \tau_2 \#rg\)-closed, \( f(X \setminus V) \) is \( \tau_1 \tau_2 \#rg\)-closed. Since \( f \) is a bijection, \( Y \setminus f(V) \) is \( \tau_1 \tau_2 \#rg\)-closed that implies \( f(V) \) is \( \tau_1 \tau_2 \#rg\)-open in \( Y \). Since \( g = f_1^{-1} \) and since \( g \) and \( f \) are bijection \( g^{-1}(V) = f(V) \) so that \( g^{-1}(V) \) is \( \tau_1 \tau_2 \#rg\)-open in \( Y \). Therefore \( f_1^{-1} \) is \( \tau_1 \tau_2 \#rg\)-irresolute. This proves (2) \( \Rightarrow \) (3).

Suppose \( f_1^{-1} \) is \( \tau_1 \tau_2 \#rg\)-irresolute. Let \( V \) be \( \tau_1 \tau_2 \#rg\)-open in \( X \). Since \( f_1^{-1} \) is \( \tau_1 \tau_2 \#rg\)-irresolute and \( f^{-1}(X \setminus V) = f(X \setminus V) = Y \setminus f(V) \) is \( \tau_1 \tau_2 \#rg\)-closed in \( Y \) that implies \( f(V) \) is \( \tau_1 \tau_2 \#rg\)-open in \( Y \). Therefore \( f_1 \) is \( \tau_1 \tau_2 \#rg\)-open.

This proves (3) \( \Rightarrow \) (1).

Theorem 4.10 Let \( f : X \to Y \) and \( g : Y \to Z \) be two functions. Suppose \( f \) and \( g \) are \( \tau_1 \tau_2 \#rg\)-closed (resp. \( \tau_1 \tau_2 \#rg\)-open). Then \( g \circ f \) is \( \tau_1 \tau_2 \#rg\)-closed (resp. \( \tau_1 \tau_2 \#rg\)-open).

**Proof** Let \( U \) be any \( \tau_1 \tau_2 \#rg\)-closed (resp. \( \tau_1 \tau_2 \#rg\)-open) set in \( X \). Since \( f \) is \( \tau_1 \tau_2 \#rg\)-closed, using Definition 4.8, \( f(U) \) is \( \tau_1 \tau_2 \#rg\)-closed (resp. \( \tau_1 \tau_2 \#rg\)-open) in \( Y \). Again since \( g \) is \( \tau_1 \tau_2 \#rg\)-closed (resp. \( \tau_1 \tau_2 \#rg\)-open), using Definition 4.8, \( g(f(U)) \) is \( \tau_1 \tau_2 \#rg\)-closed (resp. \( \tau_1 \tau_2 \#rg\)-open) in \( Z \). This shows that \( g \circ f \) is \( \tau_1 \tau_2 \#rg\)-closed (resp. \( \tau_1 \tau_2 \#rg\)-open).

References