Harmonic Mean Labeling Of H-Super Subdivision of Y-Tree

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Abstract

A graph $G$ with $p$ vertices and $q$ edges is called a harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from \{1,2,…..q+1\} in such a way that each edge $e = uv$ is labeled with $f(uv) = \frac{2f(u)f(v)}{f(u)+f(v)}$ (or) $\frac{2f(u)f(v)}{f(v)+f(u)}$ then the edge labels are distinct. In this case $f$ is called Harmonic mean labeling of $G$. In this paper we prove that some families of graphs such as H- super subdivision of Y-Tree HSS($Y_{n+1}$), HSS($Y_{n+1}$) $\bigodot K_1$, HSS($Y_{n+1}$) $\bigodot K_2$, HSS($Y_{n+1}$) $\bigodot K_2$ are harmonic mean graphs.

Keywords:
Harmonic mean graph, H- super subdivision of HSS($Y_{n+1}$), HSS($Y_{n+1}$) $\bigodot K_1$, HSS($Y_{n+1}$) $\bigodot K_2$, HSS($Y_{n+1}$) $\bigodot K_2$

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Introduction

Let $G=(V,E)$ be a (p,q) graph with $p = |V(G)|$ vertices and $q =|E(G)|$ edges, where $V(G)$ and $E(G)$ respectively denote the vertex set and edge set of the graph $G$. In this paper, we consider the graphs which are simple, finite and undirected. For graph theoretic terminology and notations we refer to Harary [4]

The concept of graph labeling was introduced by Rosa in 1967. A detailed survey of graph labeling is available in Gallian[3].The concept of Harmonic mean labeling of graph was introduced by S.Somasundaram, R.Ponraj and S.S.Sandhya and they investigated the existence of harmonic mean labeling of several family of graphs .The concept
We have proved Harmonic mean labeling of subdivision graphs such as $P_n \cup K_1$, $P_n \cup K_2$, H-graph, crown, $C_n \cup K_1$, $C_n \cup K_2$, quadrilateral snake, Triangular snake and also proved Harmonic mean labeling of some graphs such as Triple triangular snake $T(T_n)$, Alternate Triple triangular snake $A[T(T_n)]$, Quadrilateral snake $T(Q_n)$, Alternate Triple quadrilateral snake $A[T(Q_n)]$, Twig graph $T(n)$, Balloon triangular snake $T_n(C_m)$, key graph $K_y(m,n)$, zig-zag triangle $Z(T_n)$, $Z(T_n)$ $\cup K_1$, $Z(T_n)$ $\cup K_2$, $Z(T_n)$ $\cup K_2$, alternate zig-zag triangle $A[Z(T_n)]$, spiked snake graph $SS(4,n)$ and harmonic mean labeling of $h$-super subdivision of path, cycle graphs. The following definitions are useful for the present investigation.

**Definition: 1.1 [8]**

A graph $G = (V, E)$ with $p$ vertices and $q$ edges is called a Harmonic mean graph if it is possible to label the vertices $v \in V$ with distinct labels $f(v)$ from $\{1, 2, \ldots, q+1\}$ in such a way that when each edge $e = uv$ is labeled with $f(uv)$

$$[2f(u)f(v)] / [f(u)+f(v)]$$

then the resulting edge labels are distinct. In this case $f$ is called Harmonic mean labeling of $G$.

**Definition: 1.2 [2]**

Let $G$ be a $(p, q)$ graph. A graph obtained from $G$ by replacing each edge $e_i$ by an H-graph in such a way that the ends $e_i$ are merged with a pendent vertex in $P_2$ and a pendent vertex $P_2'$ is called H-Super Subdivision of $G$ and it is denoted by $HSS(G)$ where the H-graph is a tree on 6 vertices in which exactly two vertices of degree 3.

**Definition: 1.2 [2]**

Let $Y_{n+1}$ be a Y-tree $(n \geq 2)$ with $n+2$ vertices and $n+1$ edges. Let the vertices of $Y_{n+1}$ be $v_1, v_2, v_3, \ldots, v_{n+1}, u$. The $HSS(Y_{n+1})$ is constructed from $Y_{n+1}$ by replacing each edge by the H-graph. The vertex and edge sets of $HSS(Y_{n+1})$ are as follows

$$V(HSS(Y_{n+1})) = \{ u, v_n u(1), v_n u(2), uv_n(1), uv_n(2), v_{n+1} \}$$

$$\cup \{ v_i u v_{i+1}(1) \cup v_{i+1} v_{i+1}(1) u v_{i+2}(1) / 1 \leq i \leq n \}$$

E (HSS($Y_{n+1}$)) = $E_1 \cup E_2$ where

$$E_1 = \{ v_n u(1), v_n u(2), v_n u(1) uv_n(1), uv_n(1), uv_n(2), uv_n(1) u \},$$

$$E_2 = \{ v_i v_{i+1}(1), v_{i+1}(1) v_{i+2}(1), v_{i+1}(1) v_{i+1}(1) v_{i+1}(1), v_{i+1}(1) v_{i+1}(1), v_{i+1}(1) / 1 \leq i \leq n \}$$

Then $HSS(Y_{n+1})$ has $5n+6$ vertices and $5n+5$ edges.
In this paper we prove that $H$-super subdivision of $HSS(Y_{n+1})$, $HSS(Y_{n+1}) \circ K_1$, $HSS(Y_{n+1}) \circ K_2$, $HSS(Y_{n+1}) \circ K_2$ are harmonic mean graph.

II. Harmonic mean labeling of graphs

Theorem 2.1

The structure of $H$-super subdivision of Y-tree $HSS(Y_{n+1})$ is a harmonic mean graph.

Proof:

Let $HSS(Y_{n+1})$ be the $H$-super subdivision of a Y-tree $Y_{n+1}$ which has $5n+6$ vertices and $5n+5$ edges. The vertex set

$V(G) = \{ u, u_{n}^{(1)}, u_{n}^{(2)}, v_{n}^{(1)}, v_{n}^{(2)}, v_{n+1}\} \cup \{ v_{i} \cup x_{i} \cup r_{i} \cup s_{i} \cup y_{i} \mid 1 \leq i \leq n \}$

and the edge set.

$E(G) = \{ v_{n} v_{n}^{(2)}, v_{n} u_{n}^{(2)}, v_{n} u_{n}^{(1)}, v_{n} u_{n}^{(2)}, v_{n} u_{n}^{(1)}, v_{n} u_{n}^{(1)}, v_{n} u_{n}^{(2)} \} \cup \{ v_{i} x_{i}, x_{i} y_{i}, x_{i} y_{i}, s_{i} y_{i}, y_{i} v_{i+1} \mid 1 \leq i \leq n \}$.

Then the resultant graph is harmonic mean labeling of structure of $H$-super subdivision of Y-tree graph.

Define a function $f : V(G) \rightarrow \{1, 2, \ldots, q + 1\}$ by

\[
\begin{align*}
    f(v_{i}) & = 5i - 4 & \text{for } 1 \leq i \leq n \\
    f(x_{i}) & = 5i - 3 & \text{for } 1 \leq i \leq n \\
    f(y_{i}) & = 5i & \text{for } 1 \leq i \leq n \\
    f(r_{i}) & = 5i - 2 & \text{for } 1 \leq i \leq n \\
    f(s_{i}) & = 5i - 1 & \text{for } 1 \leq i \leq n \\
    f(u) & = 5n + 2 \\
    f(u_{n}^{(1)}) & = 5n + 3 \\
    f(u_{n}^{(2)}) & = 5n + 4
\end{align*}
\]
\[ f(v_n u^{(1)}) = 5n + 5 \]
\[ f(v_n u^{(2)}) = 5n + 6 \]

Then the resulting edge labels are distinct.

\[ f(v_i x_i) = 5i - 4 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(x_i y_i) = 5i - 3 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(x_i y_i) = 5i - 2 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(s_i y_i) = 5i - 1 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(y_i v_{i+1}) = 5i \]
\[ f(v_n v_n u^{(2)}) = 5n \]
\[ f(u v_n^{(1)}) = 5n + 2 \]
\[ f(u v_n^{(1)} u v_n^{(2)}) = 5n + 3 \]
\[ f(u v_n^{(1)} v_n u^{(2)}) = 5n + 4 \]
\[ f(v_n u^{(1)} v_n u^{(2)}) = 5n + 5 \]

Thus \( f \) provides a harmonic mean labeling of graph \( G \).

Hence \( G \) is a harmonic mean graph.

**Example: 2.1.1**

A harmonic mean labeling of graph \( G \) obtained by structure of \( H \)- super subdivision of \( Y \)-tree \( \text{HSS}(Y_{n+1}) \) are shown in fig 2.1.1

**Theorem: 2.2**

The structure of \( H \)- super subdivision of \( Y \)-tree \( \text{HSS}(Y_{n+1}) \bigcirc K_1 \) is a harmonic mean graph.
Proof:

Let \( HSS(Y_{n+1}) \) be the H- super subdivision of a Y-tree \( Y_{n+1} \) which has \( 5n+6 \) vertices and 
\( 5n+5 \) edges and every vertex attached by \( K_1 \) graph. Then the resultant graph is \( HSS(Y_{n+1}) \odot K_1 \) graph whose vertex set

\[
V(G) = \{ z, zw_1, zw_2, w_1z, w_2z, k_1, k_2, k_3, l_1, l_2, u_{n+1}, v_{n+1}, w_{n+1}, s_{n+1} \} \cup \{ u_i, v_i, w_i, s_i, r_i, t_i, x_i, y_i/ 1 \leq i \leq n \} \\
U \{ p_i, q_i / 1 \leq i \leq n - 1 \}.
\]

And the edge set

\[
E(G) = \{ zzw_1, zzw_2, zw_1k_1, zw_2k_2, zw_1zw_2, w_1zkw_2, k_2w_1z, k_3zw_1, k_3zw_2, w_1zw_2, w_1zw_2l_1, w_1zw_2l_2, v_iu_i, v_iw_i, u_{i+1}w_{i+1}, v_is_i, v_ir_i, \\
t_{i+1}u_i, r_ix_i, t_iy_i, / 1 \leq i \leq n \} \cup \{ p_{i+1}u_i, q_{i+1}v_{i+1} / 1 \leq i \leq n - 1 \} \cup \{ u_1v_1, u_{n+1}v_{n+1}, u_{n+1}w_{n+1}, w_{n+1}s_{n+1} \}.
\]

Then the resultant graph is harmonic mean labeling of structure of H- super subdivision of \( HSS(Y_{n+1}) \odot K_1 \) Y-tree graph.

Define a function \( f: V(G) \rightarrow \{1, 2, \ldots, q + 1\} \) by

\[
\begin{align*}
f(u_1) &= 1 \\
f(u_i) &= 10i - 11 \quad \text{for} \ 2 \leq i \leq n + 1 \\
f(v_i) &= 10i - 6 \quad \text{for} \ 1 \leq i \leq n \\
f(v_{n+1}) &= 10n + 12 \\
f(w_1) &= 3 \\
f(w_i) &= 10i - 8 \quad \text{for} \ 2 \leq i \leq n \\
f(w_{n+1}) &= 10n \\
f(p_i) &= 10i \quad \text{for} \ 1 \leq i \leq n - 1 \\
f(q_i) &= 10i + 3 \quad \text{for} \ 1 \leq i \leq n - 1 \\
f(s_1) &= 2 \\
f(s_i) &= 10i - 9 \quad \text{for} \ 2 \leq i \leq n + 1 \\
f(r_i) &= 10i - 5 \quad \text{for} \ 1 \leq i \leq n \\
f(t_i) &= 10i - 2 \quad \text{for} \ 1 \leq i \leq n \\
f(x_i) &= 10i - 4 \quad \text{for} \ 1 \leq i \leq n \\
f(y_i) &= 10i - 3 \quad \text{for} \ 1 \leq i \leq n \\
f(z) &= 10n + 3 \\
f(zw_1) &= 10n + 5
\end{align*}
\]
Then the resulting edge labels are distinct.

\[
\begin{align*}
  f(u_1v_1) &= 1 \\
  f(v_1u_{i+1}) &= 10i - 4 & \text{for } 1 \leq i \leq n \\
  f(u_{n+1}w_{n+1}) &= 10n - 1 \\
  f(p_iu_{i+1}) &= 10i - 1 & \text{for } 1 \leq i \leq n-1 \\
  f(q_iw_{i+1}) &= 10i + 3 & \text{for } 1 \leq i \leq n-1 \\
  f(s_1w_1) &= 2 \\
  f(s_1w_i) &= 10i - 9 & \text{for } 2 \leq i \leq n + 1 \\
  f(r_iu_i) &= 10i - 6 & \text{for } 1 \leq i \leq n \\
  f(t_iu_{i+1}) &= 10i - 2 & \text{for } 1 \leq i \leq n \\
  f(r_ix_i) &= 10i - 5 & \text{for } 1 \leq i \leq n \\
  f(t_iy_i) &= 10i - 3 & \text{for } 1 \leq i \leq n \\
  f(zk_{n(1)}) &= 10n + 2 \\
  f(zzw_{n(1)}) &= 10n + 3 \\
  f(zw_{n(1)}kk_{n(2)}) &= 10n + 4 \\
  f(zw_{n(2)}zw_{n(2)}) &= 10n + 6 \\
  f(zw_{n(2)}kk_{n(3)}) &= 10n + 7 \\
  f(zw_{n(1)}zw_{n(2)}) &= 10n + 8 \\
  f(wzw_{n(1)}w_{n(2)}) &= 10n + 9 \\
\end{align*}
\]
Thus \( f \) provides a harmonic mean labeling of graph \( G \).
Hence \( G \) is a harmonic mean graph.

**Example: 2.2.1**

A harmonic mean labeling of graph \( G \) obtained by structure of \( H\)-super subdivision of \( Y\)-tree \( HSS(Y_{n+1}) \circ K_1 \) are shown in fig 2.2.1

**Theorem: 2.3**

The structure of \( H\)-super subdivision of \( Y\)-tree \( HSS(Y_{n+1}) \circ K_2 \) is a harmonic mean graph.

**Proof:**

Let \( HSS(Y_{n+1}) \) be the \( H\)-super subdivision of a \( Y\)-tree \( Y_{n+1} \) which has \( 5n+6 \) vertices and \( 5n+5 \) edges and every vertex attached by \( K_2 \) graph. Then the resultant graph is \( HSS(Y_{n+1}) \circ K_2 \) graph whose vertex set

\[
V(G) = \{ \{ u_i, v_i, x_i, y_i, z_i, p_i, q_i, r_i, s_i, t_i, e_i, d_i, f_i, k_i, l_i, / 1 \leq i \leq n \} \} \cup 
\{ z_{n+1}, s_{n+1}, r_{n+1}, w, w z_n^{(1)}, w z_n^{(2)}, z_n w^{(1)}, z_n w^{(2)}, 
\quad g_n^{(1)}, g_n^{(2)}, g_n^{(3)}, g_n^{(4)}, g_n^{(5)}, g_n^{(6)}, h_n^{(1)}, h_n^{(2)}, h_n^{(3)}, h_n^{(4)} \}.
\]

And the edge set

\[
E(G) = \{ \{ w g_n^{(1)}, w g_n^{(2)}, w z_n^{(1)}, w z_n^{(2)}, g_n^{(1)}, g_n^{(2)}, g_n^{(3)}, g_n^{(4)}, w z_n^{(2)}, g_n^{(5)}, w z_n^{(2)}, g_n^{(6)}, w w z_n^{(1)}, w z_n^{(1)}, w z_n^{(2)}, w z_n^{(1)}, z_n w^{2}, z_n w^{(1)}, h_n^{(1)}, z_n w^{(1)} h_n^{(2)}, 
\quad z_n w^{(2)} h_n^{(3)}, z_n w^{(2)} h_n^{(4)}, z_n w^{(1)} z_n w^{(2)}, z_n w^{(2)} z_n \} \cup \{ u_i x_i, v_i x_i, x_i y_i, z_i x_i, 
\quad y_i z_{i+1}, z_i r_i, z_i s_i, x_i t_i, y_i e_i, t_i d_i, t_i f_i, e_i k_i, e_i l_i / 1 \leq i \leq n \} \cup \{ y_{n} z_{n+1}, 
\quad z_{n+1} s_{n+1}, z_{n+1} r_{n+1} \}.
\]
Then the resultant graph is harmonic mean labeling of structure of H- super subdivision of $HSS(Y_{n+1} \circ\overline{K}_2)$ Y-tree graph.

Define a function $f : V(G) \rightarrow \{1,2,\ldots,q + 1\}$ by

- $f(x_i) = 15i-9$ for $1 \leq i \leq n$
- $f(y_i) = 15i-1$ for $1 \leq i \leq n$
- $f(u_1) = 1$
- $f(u_i) = 15i-11$ for $2 \leq i \leq n$
- $f(v_i) = 15i-10$ for $1 \leq i \leq n$
- $f(p_i) = 15i-2$ for $1 \leq i \leq n$
- $f(q_i) = 15i$ for $1 \leq i \leq n$
- $f(z_1) = 4$
- $f(z_i) = 15i-14$ for $2 \leq i \leq n-1$
- $f(z_n+1) = 15n+3$
- $f(r_i) = 15i-13$ for $1 \leq i \leq n+1$
- $f(s_i) = 15i-12$ for $1 \leq i \leq n-1$
- $f(s_{n+1}) = 15n+1$
- $f(t_i) = 15i-8$ for $1 \leq i \leq n$
- $f(e_i) = 15i-4$ for $1 \leq i \leq n$
- $f(d_i) = 15i-7$ for $1 \leq i \leq n$
- $f(f_i) = 15i-6$ for $1 \leq i \leq n$
- $f(k_i) = 15i-5$ for $1 \leq i \leq n$
- $f(l_i) = 15i-3$ for $1 \leq i \leq n$
- $f(w) = 15n+5$
- $f(g_n^{(1)}) = 15n+6$
- $f(g_n^{(2)}) = 15n+4$
- $f(g_n^{(3)}) = 15n+9$
- $f(g_n^{(4)}) = 15n+7$
- $f(g_n^{(5)}) = 15n+12$
- $f(g_n^{(6)}) = 15n+10$
- $f(w z_n^{(1)}) = 15n+8$
- $f(w z_n^{(2)}) = 15n+11$
- $f(z_n w^{(1)}) = 15n+14$
\[ f(z_n w^{(2)}) = 15n + 17 \]
\[ f(h_n^{(1)}) = 15n + 13 \]
\[ f(h_n^{(2)}) = 15n + 15 \]
\[ f(h_n^{(3)}) = 15n + 16 \]
\[ f(h_n^{(4)}) = 15n + 18 \]

Then the resulting edge labels are distinct.

\[ f(x_i y_i) = 15i - 6 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(x_1 z_1) = 4 \]
\[ f(x_i z_i) = 15i - 12 \quad \text{for} \quad 2 \leq i \leq n \]
\[ f(y_i z_{i+1}) = 15i \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(y_n z_{n+1}) = 15n + 1 \]
\[ f(x_1 u_1) = 1 \]
\[ f(x_i u_i) = 15i - 11 \quad \text{for} \quad 2 \leq i \leq n \]
\[ f(x_i v_i) = 15i - 10 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(y_i p_i) = 15i - 2 \quad \text{for} \quad 1 \leq i \leq n - 1 \]
\[ f(y_n p_n) = 15n - 1 \quad \text{for} \quad 1 \leq i \leq n - 1 \]
\[ f(y_i q_i) = 15i - 1 \quad \text{for} \quad 1 \leq i \leq n - 1 \]
\[ f(y_n p_n) = 15n \]
\[ f(z_1 r_1) = 2 \]
\[ f(z_i r_i) = 15i - 14 \quad \text{for} \quad 2 \leq i \leq n \]
\[ f(z_{n+1} r_{n+1}) = 15n + 3 \]
\[ f(z_1 s_1) = 3 \]
\[ f(z_i s_i) = 15i - 13 \quad \text{for} \quad 2 \leq i \leq n \]
\[ f(x_i t_i) = 15i - 9 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(y_i e_i) = 15i - 3 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(t_i d_i) = 15i - 8 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(t_i f_i) = 15i - 7 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(e_i k_i) = 15i - 5 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(e_i l_i) = 15i - 4 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(z_n z_n w^{(2)}) = 15n - 2 \]
\[ f(w g_n^{(1)}) = 15n + 5 \]
Thus \( f \) provides a harmonic mean labeling of graph \( G \).
Hence \( G \) is a harmonic mean graph.

**Example: 2.3.1**

A harmonic mean labeling of graph \( G \) obtained by structure of \( H \)-super subdivision of Y-tree \( HSS(Y_{n+1}) \odot K_2 \) are shown in fig 2.3.1.

**Theorem: 2.4**

The structure of \( H \)-super subdivision of Y-tree \( HSS(Y_{n+1}) \odot K_2 \) is a harmonic mean graph.
Proof:

Let HSS\( (Y_{n+1}) \) be the H- super subdivision of a Y-tree \( Y_{n+1} \) which has 5n+6 vertices and 5n+5 edges and every vertex attached by \( K_2 \) graph. Then the resultant graph is \( HSS(Y_{n+1}) \bigodot K_2 \)
graph whose vertex set

\[
V(G) = \{ \{u_i, v_i, x_i, y_i, z_i, p_i, q_i, r_i, s_i, t_i, e_i, d_i, f_i, k_i, l_i / 1 \leq i \leq n \} \} \cup \{p_i, q_i / 1 \leq i \leq n-1\} \\
[ \{z_{n+1}, s_{n+1}, r_{n+1}, w, w z_n^{(1)}, w z_n^{(2)}, z_n w^{(1)}, z_n w^{(2)}, \}
[ g_n^{(1)}, g_n^{(2)}, g_n^{(3)}, g_n^{(4)}, g_n^{(5)}, g_n^{(6)}, h_n^{(1)}, h_n^{(2)}, h_n^{(3)}, h_n^{(4)} \} \}
\]

And the edge set

\[
E(G) = \{ 1 \}
[ \{w g_n^{(1)}, w g_n^{(2)}, g_n^{(1)}, g_n^{(2)}, w w z_n^{(1)}, w z_n^{(1)}, w z_n^{(2)}, g_n^{(3)}, g_n^{(4)}, \}
[ w z_n^{(2)}, w z_n^{(1)}, w z_n^{(2)}, g_n^{(5)}, w z_n^{(2)}, g_n^{(6)}, g_n^{(5)}, g_n^{(6)}, w z_n^{(1)}, z_n w^{(2)}, \}
\]

Then the resultant graph is harmonic mean labeling of structure of H- super subdivision of \( HSS(Y_{n+1}) \bigodot K_2 \) Y-tree graph.

Define a function \( f : V(G) \rightarrow \{1, 2, \ldots, q + 1\} \) by

\[
\begin{align*}
 f(x_1) &= 8 \\
 f(x_i) &= 20i - 13 \quad \text{for } 2 \leq i \leq n \\
 f(y_i) &= 20i - 1 \quad \text{for } 1 \leq i \leq n \\
 f(z_1) &= 3 \\
 f(z_i) &= 20i - 19 \quad \text{for } 2 \leq i \leq n \\
 f(z_{n+1}) &= 20n + 3 \\
 f(u_i) &= 20i - 15 \quad \text{for } 1 \leq i \leq n \\
 f(v_1) &= 7 \\
 f(v_i) &= 20i - 14 \quad \text{for } 2 \leq i \leq n \\
 f(p_i) &= 20i - 3 \quad \text{for } 1 \leq i \leq n - 1 \\
 f(p_n) &= 20n - 4 \\
 f(q_i) &= 20i - 2 \quad \text{for } 1 \leq i \leq n - 1 \\
 f(q_n) &= 20n + 1 \\
 f(r_1) &= 1 \\
 f(r_i) &= 20i - 18 \quad \text{for } 2 \leq i \leq n 
\end{align*}
\]
Then the resulting edge labels are distinct.

\[
\begin{align*}
  f(x_i y_i) &= 20i - 8 & \text{for } 1 \leq i \leq n \\
  f(x_i z_i) &= 20i - 16 & \text{for } 1 \leq i \leq n \\
  f(y_i z_{i+1}) &= 20i & \text{for } 1 \leq i \leq n \\
  f(y_n z_{n+1}) &= 20n+1 \\
  f(x_i u_i) &= 20i - 14 & \text{for } 1 \leq i \leq n \\
  f(x_i v_i) &= 20i - 13 & \text{for } 1 \leq i \leq n 
\end{align*}
\]
\[ f(u_i v_i) = 20i - 15 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(y_i p_i) = 20i - 2 \quad \text{for} \quad 1 \leq i \leq n-1 \]
\[ f(y_n p_n) = 20n - 3 \]
\[ f(y_i q_i) = 20i - 1 \quad \text{for} \quad 1 \leq i \leq n-1 \]
\[ f(y_n q_n) = 20n \]
\[ f(p_i q_i) = 20i - 3 \quad \text{for} \quad 1 \leq i \leq n-1 \]
\[ f(p_n q_n) = 20n - 1 \]
\[ f(z_i r_i) = 20i - 19 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(z_{n+1} r_{n+1}) = 20n + 2 \]
\[ f(z_1 s_1) = 3 \]
\[ f(z_i s_i) = 20i - 18 \quad \text{for} \quad 2 \leq i \leq n \]
\[ f(z_{n+1} s_{n+1}) = 20n + 4 \]
\[ f(r_1 s_1) = 2 \]
\[ f(r_i s_i) = 20i - 17 \quad \text{for} \quad 2 \leq i \leq n \]
\[ f(\tau_i t_i) = 20i - 12 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(y_i e_i) = 20i - 4 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(t_i d_i) = 20i - 11 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(t_i f_i) = 20i - 10 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(d_i f_i) = 20i - 9 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(e_i k_i) = 20i - 6 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(e_i l_i) = 20i - 5 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(k_i l_i) = 20i - 7 \quad \text{for} \quad 1 \leq i \leq n \]
\[ f(w g_n^{(1)}) = 20n + 7 \]
\[ f(w g_n^{(2)}) = 20n + 5 \]
\[ f(g_n^{(1)} g_n^{(2)}) = 20n + 6 \]
\[ f(w z_n^{(1)}) = 20n + 8 \]
\[ f(w z_n^{(1)} g_n^{(4)}) = 20n + 9 \]
\[ f(g_n^{(4)} g_n^{(3)}) = 20n + 10 \]
\[ f(w z_n^{(2)} g_n^{(6)}) = 20n + 13 \]
\[ f(w z_n^{(1)} w z_n^{(2)}) = 20n + 12 \]
\[ f(g_n^{(6)} g_n^{(5)}) = 20n + 14 \]
Thus \( f \) provides a harmonic mean labeling of graph \( G \).

Hence \( G \) is a harmonic mean graph.

**Example: 2.4.1**

A harmonic mean labeling of graph \( G \) obtained by structure of \( H \)-super subdivision of \( Y \)-tree \( \text{HSS}(Y_{n+1}) \bigcirc K_2 \) are shown in fig 2.4.1

**Conclusion:**

We have presented a few new results on Harmonic mean labeling of certain classes of graphs like the that \( H \)-super subdivision of \( \text{HSS}(Y_{n+1}) \), \( \text{HSS}(Y_{n+1}) \bigcirc K_1 \), \( \text{HSS}(Y_{n+1}) \bigcirc \overline{K_2} \), \( \text{HSS}(Y_{n+1}) \bigcirc K_2 \). Analogous work can be carried out for other families and in the context of different types of graph labeling techniques.
References


