Super Padovan Graceful Labeling

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Abstract: In this paper, I introduce a new concept of labeling called Super Padovan Graceful Labeling as follows: An injective function f from V(G) into \{0, p_1, p_2, ..., p_q\} is Padovan graceful if the induced edge labeling f*(xy) = |f(x)− f(y)| is a bijection onto the set \{p_1, p_2, ..., p_q\}. A graph G(p,q) which admits a Super Padovan graceful labeling is called a Super Padovan graceful graph, where p_q is the qth Padovan number in the Padovan sequence.

Existence of Super Padovan graceful labeling of some of the graphs are discussed here.


1. INTRODUCTION

Graph labeling plays a important role in Graph theory. It is used in many applications like coding theory, x-ray crystallography and circuit design. Rosa introduced the concept of graceful labeling f of a (p,q) graph G as follows: f is a graceful labeling if f is an injection from V(G) to the set \{0,1,2,...,q\} such that when each edge uv is assigned the label |f(u)− f(v)|, the resulting edge labels are distinct. A Fibonacci graceful labeling and Super Fibonacci graceful labeling have been introduced by Kathiresan and Amutha [4] in 2006. As an extension to Fibonacci graceful labeling, we introduce Padovan graceful labeling here. Padovan graceful labeling of certain families of graphs are discussed.

2. Definitions

Definition 2.1: Let G(p,q) be a graph. A injective function f from V(G) into \{0,1,2,...,F_q\}, where F_q is the qth Fibonacci number is said to be Fibonacci graceful if the induced edge labeling f * (uv) = |f(u)− f(v)| is a bijection onto the set \{F_1, F_2, ..., F_q\}. If a graph G(p,q) admits a Fibonacci graceful labeling then G is called a Fibonacci graceful graph.

Definition 2.2: Let G(p,q) be a graph. An injective function f from V(G) into \{0, p_1, p_2, ..., p_q\}, where p_q is the qth Padovan number in the Padovan sequence is said to be Super Padovan graceful if the induced edge labeling f * (uv) = |f(u)− f(v)| is a bijection onto the set \{p_1, p_2, ..., p_q\}. If a graph G(p,q) admits a Super Padovan graceful labeling then G is called a Super Padovan graceful graph. We fix the values to be p_0 = 0, p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 4 and p_n = p_{n−2} + p_{n−3} for all n ≥ 5 (i.e) \{0,1,2,3,4,5,7,9,12,16,21,28,........\} is a Padovan sequence.

Example 2.1: In Figure 2.1 Provides an example of Super Padovan graceful labeling of a graph Figure 2.1
3. Main results

**Theorem 3.1:** the cycle $c_3, c_4, c_6$ are a Super Padovan graceful graph.

**proof:**

a) Cycle $C_3$: We assume by definition then there exists an injective function $f : V(G) \rightarrow \{0, p_1, p_2, p_3\}$ since there are 3 edges we assume $q=3$ and hence $p_q = p_3$ such that the induced edge labels are padovan numbers are \{p_1, p_2, p_3\} = \{1,2,3\}.

Let $x,y,z$ be the vertices of the cycle $C_3$, then $f(x) = 0$, $f(y) = p_1 = 1$ and $f(z) = p_3 = 3$.

Then for the vertices $x$ and $y$, $f^*(xy) = p_1 = 1$, for the vertices $y$ and $z$, $f^*(yz) = p_2 = 2$, which is a padovan number, for the vertices $z$ and $x$, $f^*(zx) = p_3 = 3$, which is a padovan number. Therefore $C_3$ is a Super Padovan graceful graph.

b) Cycle $C_4$: We assume by definition then there exists an injective function $f : V(G) \rightarrow \{0, p_1, p_2, p_3, p_4\}$ since there are 4 edges we assume $q=4$ and hence $p_q = p_4$ such that the induced edge labels are padovan numbers are \{p_1, p_2, p_3, p_4\} = \{1,2,3,4\}. Let $a,x,y,z$ be the vertices of the cycle $C_4$, then $f(a) = 0$, $f(x) = p_4 = 4$, $f(y) = p_1 = 1$ and $f(z) = p_2 = 2$.

Then for the vertices $a$ and $x$, $f^*(ax) = p_4 = 4$, for the vertices $x$ and $y$, $f^*(xy) = p_3 = 3$, which is a padovan number, for the vertices $y$ and $z$, $f^*(yz) = p_1 = 1$, which is a padovan number, for the vertices $z$ and $a$, $f^*(za) = p_2 = 2$, which is a padovan number. Therefore $C_4$ is a Super Padovan graceful graph.

c) Cycle $C_6$: We assume by definition then there exists an injective function $f : V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5, p_6\}$ since there are 6 edges we assume $q=6$ and hence $p_q = p_6 = 7$ such that the induced edge labels are padovan numbers are \{p_1, p_2, p_3, p_4, p_5, p_6\} = \{1,2,3,4,5,7\}. Let $a,b,c,d,e,f$ be the vertices of the cycle $C_6$, then the labeling is as follows $f(a) = 0$, $f(b) = p_6 = 7$, $f(c) = p_3 = 3$, $f(d) = p_1 = 1$, $f(e) = p_2 = 2$ and $f(f) = p_5 = 5$.

Then the resulting Super Padovan graceful labeling graph is shown below:

**Observation 3.1:**

i) A cycle to be a Super Padovan graceful labeling has an even number of odd Padovan numbers. Cycle $C_6$ has 4 odd Padovan numbers and Cycle $C_{10}$ has 6 odd Padovan numbers which is shown below:
Theorem 3.2: The Wheel $W_3$ is Super Padovan graceful labeling.

Proof: By definition there exists an injective function $f : V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5, p_6\}$ since there are 6 edges we assume $q=6$ and hence $p_q = p_6 = 7$ such that the induced edge labels are padovan numbers are \{p_1, p_2, p_3, p_4, p_5, p_6\} = \{1,2,3,4,5,7\}. Without loss of generality we assume $u,v,x,y$ are the vertices of the cycle $C_6$. Then the labeling as follows $f(u)=0, f(v)=p_6=7, f(x)=p_5=5, f(y)=p_4=4$.

- $f^*(uv)=p_6=7$
- $f^*(ux)=p_5=5$
- $f^*(uy)=p_4=4$
- $f^*(vx)=p_2=2$
- $f^*(vy)=p_3=3$
- $f^*(xy)=p_1=1$

from which we get all padovan numbers, hence $W_3$ is Super Padovan graceful labeling.

Figure 3.2

Theorem 3.3: The Path $p_2, p_3, p_4, p_5, p_6, p_7, p_8$ are all Super Padovan graceful graphs.

Proof: a) Path $p_4$: According to the definition of Padovan graceful graphs the range of labels of the vertex varies from 0 to $p_q$ where $q$ is the size of the graph and $p_q$ is the $q$th Padovan number ($V = \{0, p_1, p_2, \ldots, p_q\}$). So the order of $p_4$ is 4 and size of the graph is 3. There exists a function $f : V(G) \rightarrow \{0, p_1, p_2, p_3\}$ such that the induced edge labels are padovan numbers are $\{p_1, p_2, p_3\} = \{1,2,3\}$. Super Padovan labeling is illustrated in the figure.

b) Path $p_5$: There exists a function $f : V(G) \rightarrow \{0, p_1, p_2, p_3, p_4\}$ such that the induced edge labels are padovan numbers are $\{p_1, p_2, p_3, p_4\} = \{1,2,3,4\}$. Super Padovan labeling is illustrated in the figure.
c) Path $p_6$: There exists a function $f: V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5\}$ such that the induced edge labels are padovan numbers are $\{p_1, p_2, p_3, p_4, p_5\} = \{1, 2, 3, 4, 5\}$. Super Padovan labeling is illustrated in the figure.

![Figure 3.3.b](image)

Figure 3.3.b

d) Path $p_7$: There exists a function $f: V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5, p_6\}$ such that the induced edge labels are padovan numbers are $\{p_1, p_2, p_3, p_4, p_5, p_6\} = \{1, 2, 3, 4, 5, 7\}$. Super Padovan labeling is illustrated in the figure.

![Figure 3.3.c](image)

Figure 3.3.c

e) Path $p_8$: There exists a function $f: V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ such that the induced edge labels are padovan numbers are $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\} = \{1, 2, 3, 4, 5, 7, 9\}$. Super Padovan labeling is illustrated in the figure.

![Figure 3.3.d](image)

Figure 3.3.d

**Theorem 3.4:** Friendship graph $f_4$ is Super Padovan graceful labeling.

**Proof:** Let $f_4$ be the friendship graph. The order of $f_4$ is $p=9$ and the size of $f_4$ is $q=12$. By definition of labeling, there exists a function $f: V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ such that the induced edge labels are padovan numbers are $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\} = \{1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37\}$. Then the labeling as follows: let $f(v_0)=0$, $x_1, x_2, x_3, x_4$ are the first vertices of the triangle and $y_1, y_2, y_3, y_4$ are the second vertices of the triangle, if $f(x_1)=28$, $f(x_2)=21$, $f(x_3)=12$, $f(x_4)=3$, $f(y_1)=37$, $f(y_2)=16$, $f(y_3)=5$, $f(y_4)=1$.

$$
\begin{align*}
&f^*(v_0x_1) = 28 \\
&f^*(v_0y_1) = 37 \\
&f^*(v_0x_2) = 21 \\
&f^*(v_0y_2) = 16 \\
&f^*(v_0x_3) = 12 \\
&f^*(v_0y_3) = 5 \\
&f^*(v_0x_4) = 3 \\
&f^*(v_0y_4) = 1 \\
&f^*(x_1y_1) = 9 \\
&f^*(x_2y_2) = 4
\end{align*}
$$
\[ f^*(xy_3) = 7 \]
\[ f^*(xy_4) = 2 \]

from which we get the induced edge labels are padovan numbers are \( \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_9 \} = \{1,2,3,4,5,7,9,12,16,21,28,37\} \). Hence \( f_4 \) is a Super Padovan graceful labeling.

**Theorem 3.5:** Complete graphs \( k_n \) is Super Padovan graceful labeling iff \( n \leq 4 \).

**Proof:**

i) case \( n=3 \): We assume by definition then there exists an injective function \( f : V(G) \rightarrow \{0, p_1, p_2, p_3 \} \) since there are 3 edges we assume \( q=3 \) and hence \( p_3 \) such that the induced edge labels are padovan numbers are \( \{p_1, p_2, p_3 \} = \{1,2,3\} \).

let \( x, y, z \) be the vertices of the complete graph \( k_3 \), then \( f(x) = 0, f(y) = p_1 = 1 \) and \( f(z) = p_3 = 3 \).

then for the vertices \( x \) and \( y \), \( f^*(xy) = p_1 = 1 \),
for the vertices \( y \) and \( z \), \( f^*(yz) = p_2 = 2 \), which is a padovan number,
for the vertices \( z \) and \( x \), \( f^*(zx) = p_3 = 3 \), which is a padovan number.

therefore \( k_3 \) is a Super Padovan graceful graph.

ii) case \( n=4 \): We assume by definition then there exists an injective function \( f : V(G) \rightarrow \{0, p_1, p_2, p_3, p_4 \} \) since there are 4 edges we assume \( q=6 \) and hence \( p_6 \) such that the induced edge labels are padovan numbers are \( \{p_1, p_2, p_3, p_4, p_5, p_6 \} = \{1,2,3,4,5,7\} \).

let \( a, x, y, z \) be the vertices of the complete graph \( k_4 \), then \( f(a) = 0, f(x) = p_6 = 7 \), \( f(y) = p_5 = 5 \) and \( f(z) = p_4 = 4 \).

then for the vertices \( a \) and \( x \), \( f^*(ax) = p_6 = 7 \),
for the vertices \( x \) and \( y \), \( f^*(xy) = p_2 = 2 \), which is a padovan number,
for the vertices \( y \) and \( z \), \( f^*(yz) = p_1 = 1 \), which is a padovan number,
for the vertices \( z \) and \( a \), \( f^*(za) = p_4 = 4 \), which is a padovan number,
for the vertices \( z \) and \( x \), \( f^*(zx) = p_3 = 3 \), which is a padovan number,
for the vertices \( y \) and \( a \), \( f^*(ya) = p_5 = 5 \), which is a padovan number.

therefore \( k_4 \) is a Super Padovan graceful graph.

**Theorem 3.6:** Combs are Super Padovan graceful labeling for \( n=3,4,5,6 \).

**Proof:** Comb graph are of the form \( p_n AK_1 \). The order of the comb is \( p=2n \) and the size of the comb is \( q=2n-1 \).
let $x_1, x_2, x_3, x_4 \ldots x_n$ are the vertices of the path $p_n$ and $y_1, y_2, y_3, y_4 \ldots y_n$ are the pendent vertices attached to the path.

a) $n=3$, $p_3AK_1$: the order of the comb is 6 and the size of the comb is 5. There exists a function $f : V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5\}$ such that the induced edge labels are padovan numbers are $\{p_1, p_2, p_3, p_4, p_5\} = \{1,2,3,4,5\}$. Then the labeling as follows, we assume $f(x_1) = 0$, $f(x_2) = 4$, $f(x_3) = 2$, $f(y_1) = 5$, $f(y_2) = 1$, $f(y_3) = 3$

$\begin{align*}
    f^*(x_1y_1) &= 5 \\
    f^*(x_1x_2) &= 4 \\
    f^*(x_2y_2) &= 3 \\
    f^*(x_2x_3) &= 2 \\
    f^*(x_3y_3) &= 1,
\end{align*}$

from which we get the induced edge labels are padovan numbers are $\{1,2,3,4,5\}$. Hence comb $p_3AK_1$ is a Super Padovan graceful labeling.

b) $n=4$, $p_4AK_1$: The order of the comb is 8 and the size of the graph is 7. There exists a function $f : V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ such that the induced edge labels are padovan numbers are $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\} = \{1,2,3,4,5,7,9\}$ then the labeling as follows, we assume $f(x_1) = 0$, $f(x_2) = 7$, $f(x_3) = 3$, $f(x_4) = 4$, $f(y_1) = 9$, $f(y_2) = 2$, $f(y_3) = 5$, $f(y_4) = 1$

$\begin{align*}
    f^*(x_1y_1) &= 9 \\
    f^*(x_1x_2) &= 7 \\
    f^*(x_2y_2) &= 5 \\
    f^*(x_2x_3) &= 4 \\
    f^*(x_3y_3) &= 2 \\
    f^*(x_4y_4) &= 1,
\end{align*}$

from which we get the induced edge labels are padovan numbers are $\{1,2,3,4,5,7,9\}$. Hence comb $p_4AK_1$ is a Super Padovan graceful labeling.

Figure 3.6:a

Figure 3.6:b: comb $p_4AK_1$ is a Super Padovan graceful labeling.

Figure 3.6:comb $p_5AK_1$ is a Super Padovan graceful labeling.
Theorem 3.7: Coconut tree CT(6,n) is Super Padovan graceful labeling for all n.

Proof: Let CT(6,n) be the Coconut tree of order p= 6+n and size q=(6-1)+n . By Definition , we have \( u_1, u_2, u_3, ..., u_m \) be the vertices of the path and \( v_1, v_2, v_3, v_4, ..., v_n \) be the pendent vertices attached to the end vertex of the path. Define the labeling as follows : 

\[
\begin{align*}
  f(u_1) &= 0 \\
  f(v_i) &= p_6 + (i-1), \quad i=1,2,...,n \\
  f(u_i) &= p_5 - p_4 + p_3 - p_2 - p_1, \quad i = 2,3,4,5,6
\end{align*}
\]

then the above labeling admits Super Padovan graceful labeling, Hence Coconut tree CT(6,n) is Super Padovan graceful labeling for all n. The generalized graph for Coconut tree CT(6,n) for all n is shown in the figure.

Theorem 3.8: Olive tree O_4 is Super Padovan graceful labeling.

Proof: Olive Trees O_4 has the order \((n^2+n+2)/2\) and the size is \(n(n+1)/2\). So in O_4 we have 11 vertices and 10 edges. By definition of labeling ,There exists a function \( f : V(G) \rightarrow \{0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} \) such that the induced edge labels are padovan numbers are \( \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} = \{1,2,3,4,5,7,9,12,16,21\} \). Its proved to be O_4 is Super Padovan graceful graph from the figure.
4. Conclusion
In this paper we have shown that the cycle \((c_3, c_4, c_6)\), The Path \((p_2, p_3, p_5, p_6, p_7, p_8)\), The Wheel \((W_5)\), Friendship graph \((f_4)\),Combs, Coconut tree CT(6,n),Olive tree O_6 are Super Padovan graceful labeling.

5. References