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Random Staircase And Snakes And LaddersModels

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Abstract: The study considers the construction of two mathematical models namely Random Staircase model and Snakes and ladders model. The Random Stair case model gives the minimum number of hops required to climb a staircase with N steps if one can hop 1 step, 2 steps, …,or k steps forward or backward at a time with equal probability. This is a generalization of Gambler's Ruin problem with fortune probability equal to ½. Also a mathematical model for the Snakes and ladders game is constructed which is a special case of Upward Random Staircase model. This model can be utilized to evaluate the probability to finish the game for various movements.

***Index Terms*–Random Staircase model, Upward Staircase model , Snakes and Ladders model**

1.INTRODUCTION

Staircase model comes into play when a person is moving up a staircase having N steps and can hop either 1 step, 2 steps … or k steps forward or backward at a time with equal probability, assuming that the person is initially at one of the intermediate points between '0' and 'N'. Upward staircase model is a special case of the staircase modelwhich consider forward hops only.Here we HAVE formulated the minimum number of hops required to reach 'N' assuming that one is at step 0. From literature review we have seen that considerable efforts have been made for the development of mathematical theory of snakes and ladders game.[1],[2],[3] Here we present a more general approach for the snakes and ladders which can be applied to real life situations.

2.RANDOM STAIRCASE MODEL

Consider the movement of a person on a staircase between the points '0' and 'N' (0<N). Assume that the person is initially at one of the intermediate points between '0' and 'N' can climb 1 step or 2steps … or k steps, (0<k<N), forward or backward with equal probability, i.e. $\frac{1}{2k}$.

When the person reaches the point '0' or 'N' the motion stops and person remains there. So the states '0' and 'N' act as an absorbing barrier. Suppose the person is at one of the intermediate points between '0' and 'k' say 'r', and the person has now only the chance to take k more steps forward or backward provided that this k step in the backwards will not make the person go beyond '0'. If it happens the person will be reflected back to the original position 'r'.Here the states x=-1,-2,...,-(k-1) act as reflecting barriers. Similarly, consider that the person reaches one of the intermediate points between 'N-k' and 'N', say 's', and the person has now only the chance to take k more steps provided that this k step will not make the person go beyond 'N'. If it happens, the person will be reflected back to the original position 's'. Hence N+1, N+2, …, N+k-1 states will also act as reflecting barriers.So we can view Random Staircase model as generalization of Gambler ruin problem, fortune probability p=1/2, when k=1.

Let X_n denotes the position of the person at time t=n. Obviously $X_n$is a markov chain with state space $\Omega = \{0,1,2,3,\ldots,N\}$and is defined as

$$X_n = \begin{cases} X_{n-1} + Z_n, & k \le X_{n-1} \le N-k, \\ X_{n-1} + Z', & 0 < X_{n-1} = r < k \\ X_{n-1} + Z'', & N-k < X_{n-1} = s < N \\ 0, & X_{n-1} = 0 \\ N, & X_{n-1} = N \end{cases}$$

where Z' is random variable such that it can take values j= -r,-(r-1),…,-1,1,2,…,k with probability $\frac{1}{2\mathrm{k}}$ and 0 with probability $\frac{k-r}{2k}$,Z'' takes values j= -k,…-1,1,…,N-s with probability $\frac{1}{2\mathrm{k}}$ and 0 with probability $\frac{k-(N-s)}{2k}$. $Z_i\,'s$ are i.i.d. random variable with p.d.f

$$P(Z_i = j) = \begin{cases} \frac{1}{2\mathrm{k}}, & j = -k, \dots, -1, 1, \dots, k \\ 0, & otherwise \end{cases}$$

The transition probability matrix P is defined as

$$p_{i,j} = \frac{1}{2\mathrm{k}}, \quad 0 < i < N, k \le j \le N-k \text{ and } 0 < |i-j| \le k$$

$$p_{i,i} = \begin{cases} \frac{k-i}{2k}, & 0 < i < k \\ \frac{k-(N-i)}{2k}, & N-k < i < N \end{cases}$$

$$p_{0,0} = 1$$

$$p_{N,N} = 1$$

It is clear that the random staircase model is an irregular finite Markov Chain with all states transient but not '0' and 'N'

3.UPWARD RANDOM STAIRCASE MODEL

In this case we consider forward jumps only. Assume that the person initially at '0', can jump either 1 step or 2 steps … or k steps (0<k<N) with equal probability 1/k at a time. When the person reaches the point 'N' the motion of the person stops. Consider that the person reaches one of the intermediate points between 'N-k' and 'N', say 's', and the person has now only the chance to take k more steps provided that this k step will not make the person go beyond 'N'. If it happens the person will be reflected back to the original position 's'.

If X_n denote the position of the person at time t=n, the sequence of random variables can be characterized as

$$X_n = \begin{cases} X_{n-1} + Z_n, & 0 \le X_{n-1} \le N-k \\ X_{n-1} + Z', & N-k < X_{n-1} = s < N \\ N, & X_{n-1} = N \end{cases}$$

where Z'takes the value j=1,2,…,N-s with probability $\frac{1}{\mathrm{k}}$ and 0 with probability $\frac{k-(N-s)}{k}$ and$Z_i's$ are i.i.d. random variable following discrete uniform distribution with p.d.f

$$P(Z_i = x) = \begin{cases} \frac{1}{\mathrm{k}}, & x = 1, 2, \dots, k \\ 0, & otherwise \end{cases}$$

Obviously, the stochastic process is a Markov Chain with transition probabilities,

$$p_{i,j} = \frac{1}{\mathrm{k}}, 0 \le i < N, 0 < j \le N, 0 < j-i \le k, i < j$$

$$p_{i,i} = \frac{k-(N-s)}{k}, N-k < i < N$$

$$p_{N,N} = 1$$

In this Markov Chain the state 'N' act as an absorbing barrier so that 'N' is persistent and remaining all states are transient. The state space is $\Omega = \{0,1,2,3,\ldots,N\}$.Here N+1, N+2,…,, N+k-1 states act as reflecting barriers.

3.1Minimum number of movements to cover the game length

Our interest is to find the probability to reach N and minimum number of movements to reach N. The game length is defined as the number of movements to reach N starting from the initial point. To calculate the minimum game length, it is enough to compute $\min\{n: p_{N,0}^{n} > 0\}$. Let v_n be the probability vector at the n^{th} stage, i.e. $v_n = (v_n\,[i])_{1\times(N+1)}$ where $v_n\,[i]$=$p_{i,0}^{n}$,i=0,1,2,3,…N. It is obvious that $v_n = v_{n-1} \times P$, n=1,2,3,… assuming $v_0 = (1,0,0,\ldots,0)^T$, P is the t.p.m. Figure 1 shows the probability to reach the game length for N=100 and for k=3,4,5,6 and table 3.1 gives the minimum game length.

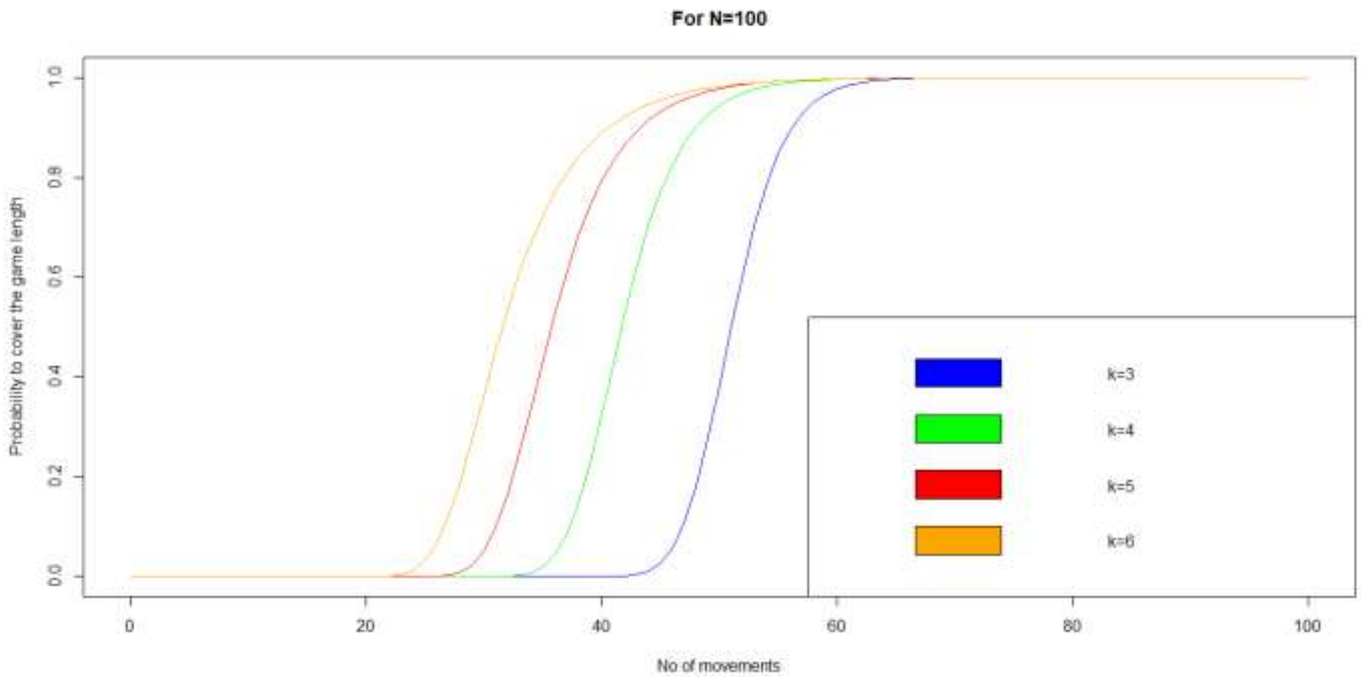

Figure1

Table 3.1

N=100	
k	Minimum game length
1	100
2	50
3	34
4	26
5	20
6	17

4.SNAKES AND LADDERS MODEL

Snakes and ladders game also known as chutes and ladders consists of 100 squares. The rules of the game and the standard board are displayed in the appendix. In the absence of snakes and ladders, the game can be viewed as an upward staircase model with N=100 and k=6. Here head of the snakes and ladders bottom positions act as reflecting barriers, i.e. whenever the pawn reaches the square which is either head of a snake or bottom of a ladder it will be reflected to the tail position of that snake or top position of the ladder. So the set consists of squares which is either head of a snake or the bottom of the ladder is not in the sample space since the pawn never visited such squares. It is to be noted that suppose the pawn is in 98, if the die shows 4 the pawn stays at 98itself, similar in the case of upward staircase model. Our main objective is to find the minimum number of movements to finish the game, i.e. minimum game length.

4.1Preliminaries

The snakes and ladders board can be mathematically represented as$\mathcal{S}[p,q]$, where p and q are the number of ladders and snakes respectively in the board. Consider the following spaces.

- A={$a_1,a_2,a_3,\ldots,a_p$} is the set consists of ladders' bottom positions.
- B={$b_1,b_2,b_3,\ldots,b_p$} is the set consists of ladders' top positions corresponds to A.
- C={$c_1,c_2,c_3,\ldots,c_p$} is the set consists of snakes' tail positions.
- D={$d_1,d_2,d_3,\ldots,d_p$} is the set of all snakes' head positions corresponds to C
- L=$\{l_i, i = 1,2, \ldots p: l_i = b_i - a_i\}$is the set of all ladders' length.
- S=$\{s_i, i = 1,2, \ldots q: s_i = d_i - c_i\}$is the set of all snakes' length.
- $\mathcal{U} = A \cup B \cup C \cup D$.It is obvious that A,B,C and D are disjoint sets.
- $\Omega = \{0,1,2,3, \ldots, 100\} - (A \cup D)$

4.2 The Model

Let us consider the movement of a pawn on the snakes and ladders board. Let X_n denote the position of the particle at the n^{th} stage, assuming $X_0 = 0$, and is defined as

$$X_n = \begin{cases} X_{n-1} + Z_n, if\ 0 \le X_{n-1} \le 94,\ X_{n-1} + Z_n \in \mathcal{U}^C \\ X_{n-1} + Z', if\ 94 < X_{n-1} = s < 100, X_{n-1} + Z_n \in \mathcal{U}^C \\ X_{n-1} + Y, if\ (X_{n-1} + Z_n) \in B \cup C \\ X_{n-1} + Y', if\ (X_{n-1} + Z_n) \in A \cup D \\ 100, if\ X_{n-1} = 100 \end{cases}$$

where$Z_i's$ are i.i.d. random variables and P$(Z_i = k) = \frac{1}{6}, k = 1,2,3 \ldots 6$ and Z' is a random variable takes values j=1,2,…,100-s with probability $\frac{1}{6}$, 0 with probability $\frac{6-(100-s)}{6}$. Let $X_{n-1} + Z_n$ =$b_k(c_k) \in$ B(C), then there exists $a_k(d_k) \in$ A(D). Y is a random variable and probability that Y takes the value Z_n :

$$P[Y = Z_n] = \begin{cases} \frac{1}{3},\ X_{n-1} < a_k(d_k) < (\le)\ X_{n-1} + 6 \\ \frac{1}{6}, otherwise \end{cases}$$

Y' is a random variable defined as :

$$Y' = \begin{cases} Z_n + l_n, (X_{n-1} + Z_n) \in A \\ Z_n - s_n, (X_{n-1} + Z_n) \in D \end{cases}$$

where:

$$P[Y' = Z_n + l_n] = \begin{cases} \frac{1}{3}, Z_n \ge l_n \\ \frac{1}{6}, otherwise \end{cases}$$

$$P[Y' = Z_n - s_n] = \begin{cases} \frac{1}{6}, s_k > Z_n \\ \frac{1}{6}, s_k = Z_n, X_{n-1} < 94 \\ \frac{7-(100-X_{n-1})}{6},\ s_k = Z_n, X_{n-1} \ge 94 \\ \frac{1}{3}, s_k < Z_n \end{cases}$$

4.2.1 Transition probability matrix

Clearly X_n depend on X_{n-1} and hence it is a Markov Chain and the transition probability matrix P is defined as:

For $\forall\ i,j \in \{0,1,2,3, \ldots, 100\}$

$$p_{i,j} = \frac{1}{6}\ , if\ 0 < j - i \leq 6\ and\ \forall\ i,j \in \mathcal{U}^C\ , \forall\ i \leq 94$$
$$p_{i,i} = \frac{6-(100-i)}{6} , \forall\ i > 94\ and\ \forall\ i \in \mathcal{U}^C$$
$$p_{i,j} = \frac{1}{3}\ , if\ 0 < j - i \leq 6, \forall\ i \in N(j)\ and\ \ \forall\ j \in B$$
$$p_{i,j} = \frac{1}{6}\ , if\ \ j - i > 6, \forall\ i \in N(j)\ and\ \ \forall\ j \in B$$
$$p_{i,j} = \frac{1}{3}\ , if\ 0 < j - i \leq 6, \forall\ i \in N(j)\ and\ \ \forall\ j \in C$$
$$p_{i,i} = \frac{1}{6}\ , \qquad \forall\ i \in N(i)\ , \forall\ i \in C, \forall\ i < 94$$
$$p_{i,i} = \frac{7-(100-i)}{6}\ , i \in N(i)\ , \forall\ i \in C, \forall\ i \geq 94$$
$$p_{i,j} = \frac{1}{6}\ , if\ \ i - j > 6, \forall\ i \in N(j)\ and\ \ \forall\ j \in C$$
$$p_{100,100} = 1$$
$$p_{i,j} = 0\ otherwise$$

Explanations:

1. Suppose we are in square 98, if the die shows 3,4,5,6 we are still in square 98 itself with probability $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{6-(100-98)}{6}$, i.e.$p_{98,98} = \frac{6-(100-98)}{6}$.
2. Suppose we are in square 67 and there is a ladder from 69 to 71. Since 69 acts as reflecting barrier, whenever the die shows 2, it immediately reflects to 71. So probability to reach 71 from 68 is increased by $\frac{1}{6}$, i.e.$p_{67,71} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. Again suppose we are in square 16 and there is a ladder from 19 to 24 then if die shows 3, we will reach to square 24. So $p_{16,24} = \frac{1}{6}$.
3. Suppose we are in square 52 there is a snake whose head is at 56 and tail is at 53. Since 56 acts as reflecting barrier, whenever the die shows 1, it immediately reflects to 53. So probability to reach 53 from 52 is increased by $\frac{1}{6}$, i.e.$p_{52,53} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. Again we are in 53 if the die shows 3 it reflects to 53 itself so $p_{53,53} = \frac{1}{6}$. Suppose we are in square 95 and if there is a snake whose head is at 98 and tail is at 95 then if the die show 3 it reflects to 95 itself, also if the die shows 6 it also reflects to 95 again. By considering these two situations $p_{95,95} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = \frac{7-(100-98)}{6}$. Suppose we are in 54 when die shows 2 reflects to 53 so $p_{54,53} = \frac{1}{6}$.

   So,
   - The state space is $\Omega = \{0,1,2,3, \ldots, 100\} - (A \cup D)$
   - Reflecting Barrier states space is $\mathcal{R}$=A ∪ D ∪ {101,102,103,104,105}
   - Absorbing Barrier states space is $\mathcal{A}$={100}
   - Transient states space is $\mathcal{T}$={0,1,2,3,…99}−($A \cup D$)
   - Persistent states space is same as $\mathcal{A}$.

4.3 Probability of finishing the game

Our prime objective is to find the probability of finishing the game, i.e. probability to reach N=100 starting initially at '0'.Let v_n be the probability vector at the n^{th} stage, i.e. $v_n = (v_n[k])_{1\times101}$ where $v_n[k]$=P[$X_n = k$],k=0,1,2,3,…100. It is obvious that $v_n = v_{n-1} \times P$, n=1,2,3,4,…, assuming $v_0 = (1,0,0, \ldots, 0)^T$ and P is the t.p.m. The minimum value of n to finish the game(minimum game length), i.e. to reach 100, is computed by min {n:$v_n[101] > 0$}.

4. ILLUSTRATIONS

Let us consider the standard chutes and ladders board shown in the appendix as an illustration

Here $\mathcal{S}[9,10]$,i.e, here there are 9 ladders and 10 snakes.
A={1,4,9,21,28,36,51,71,80}
B= {38,14,31,42,84,44,67,91,100}
C={6,26,11,53,19,60,24,73,75,78}
D={16,47,49,56,62,64,87,93,95,98}
Here the minimum number of movements to finish the gamei.e the game length is7. The movements are $0\overset{4}{\rightarrow}14\overset{6}{\rightarrow}20\overset{6}{\rightarrow}26\overset{2}{\rightarrow}84\overset{6}{\rightarrow}90\overset{6}{\rightarrow}96\overset{4}{\rightarrow}100, where \overset{n}{\rightarrow} n\ shows\ the\ number\ when\ die\ rolls.$

Figure 2 shows the the probability to finish the game

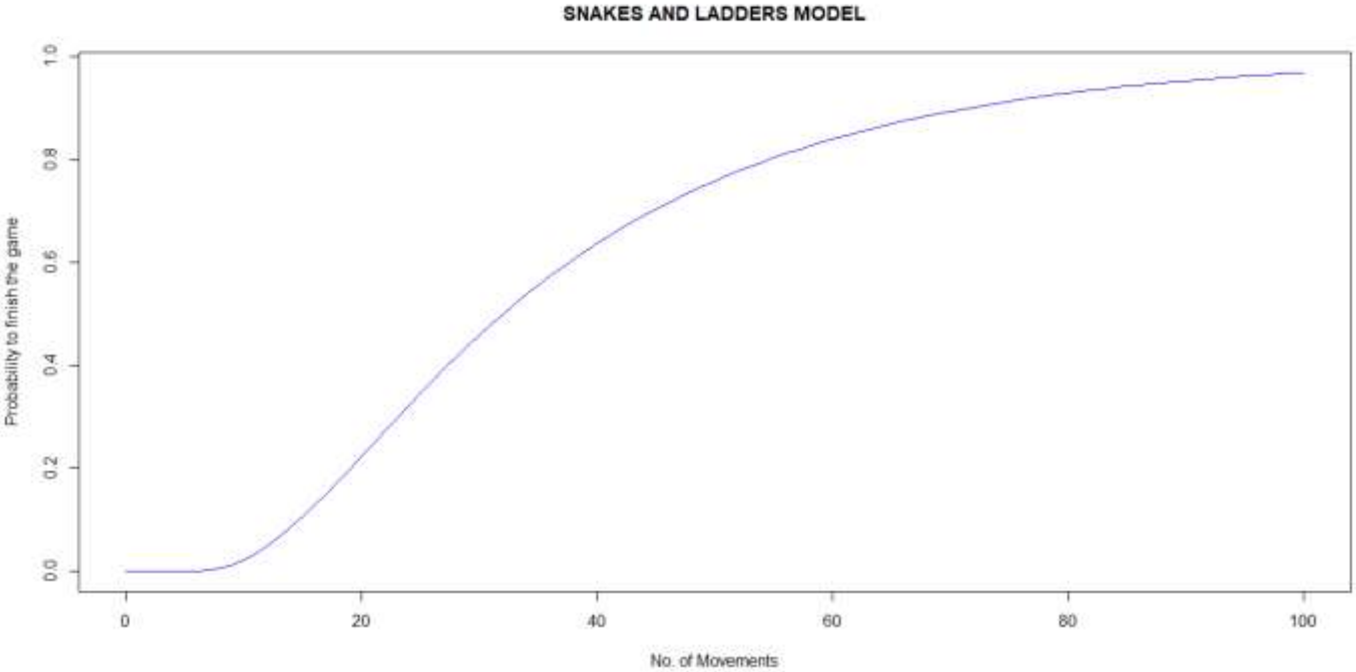

Figure 2

5.1 Special cases

Here we have tried to compare the probability to finish the game of standard chutes and ladders board with following cases
Case 1: There are no ladders
Here the minimum number of movements to finish the game is 17 and the graph is

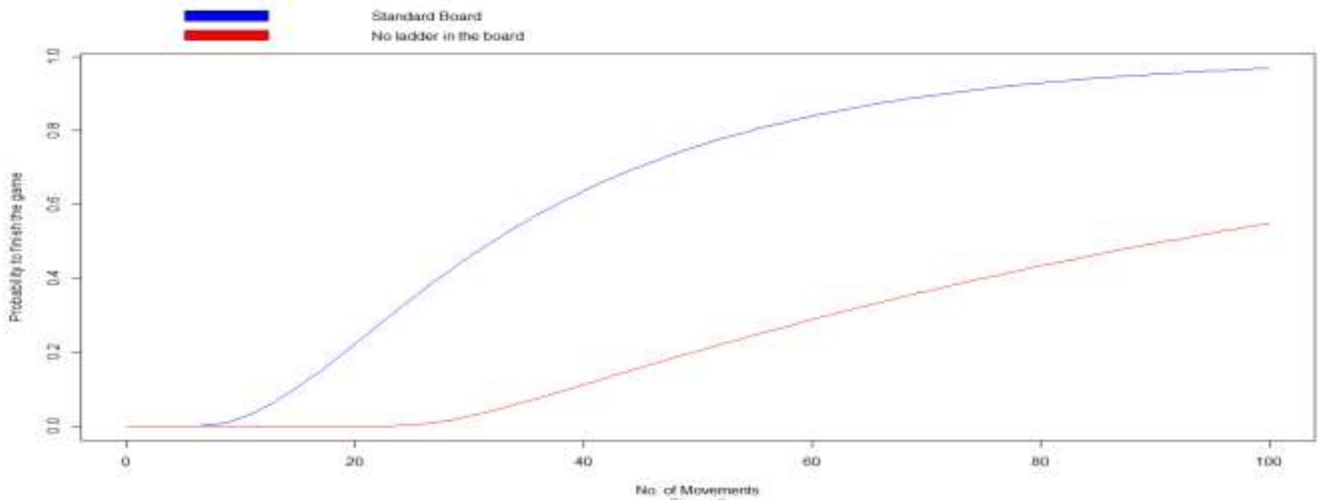

Figure 3

Clearly the probability of finishing the game is decreased compared to the standard one

Case 2: There are no snakes

Here the minimum number of movements to finish the game is 7 and the graph is shown below.

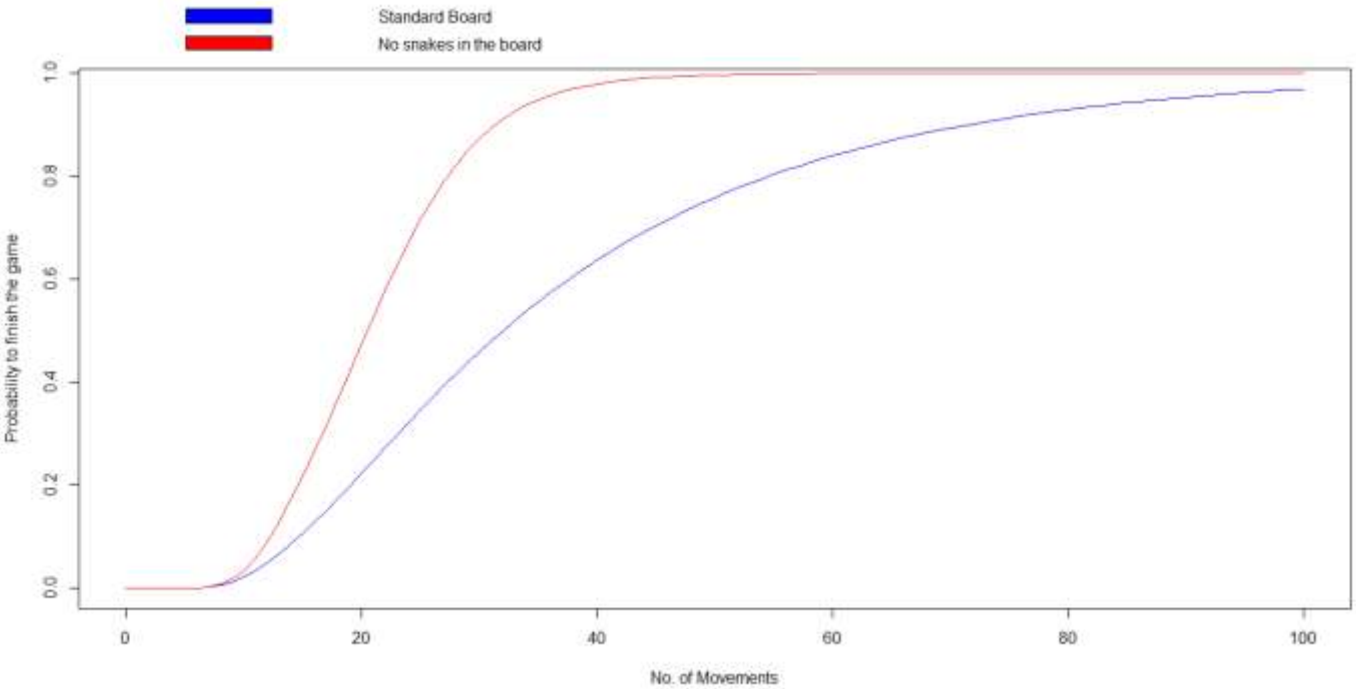

Figure 4

Clearly probability of finishing the game is increased compared to standard.

Case 3

Effect of adding a ladder to the non-ladder square in standard chutes and board. Here we add a ladder from 27 to 86. Here the minimum number of movements to finish the game is 7 and the graph is given below.

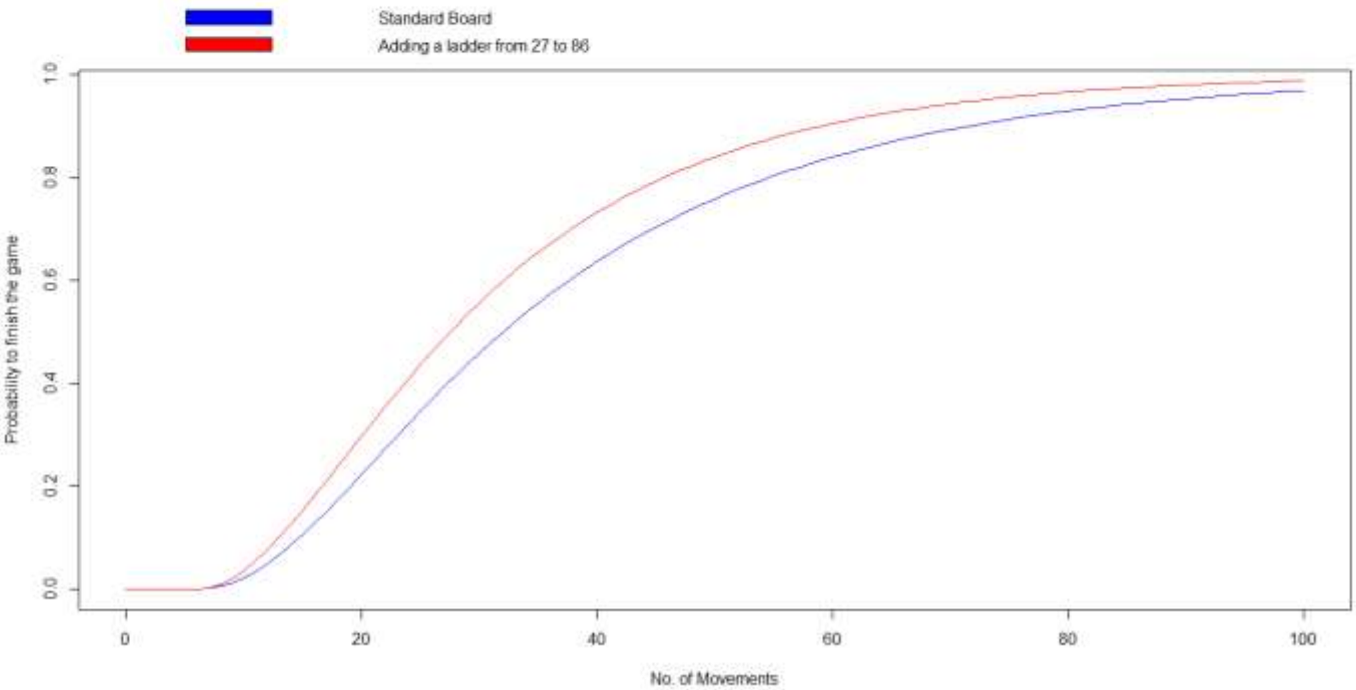

Figure 5

Clearly the probability of finishing the game is increased.

Case 4

Effect of adding a chute to the non-ladder square in standard chutes and board. Here we add a chute between 26 and 92. Here the minimum number of movements to finish the game is 8 and the graph is shown below.

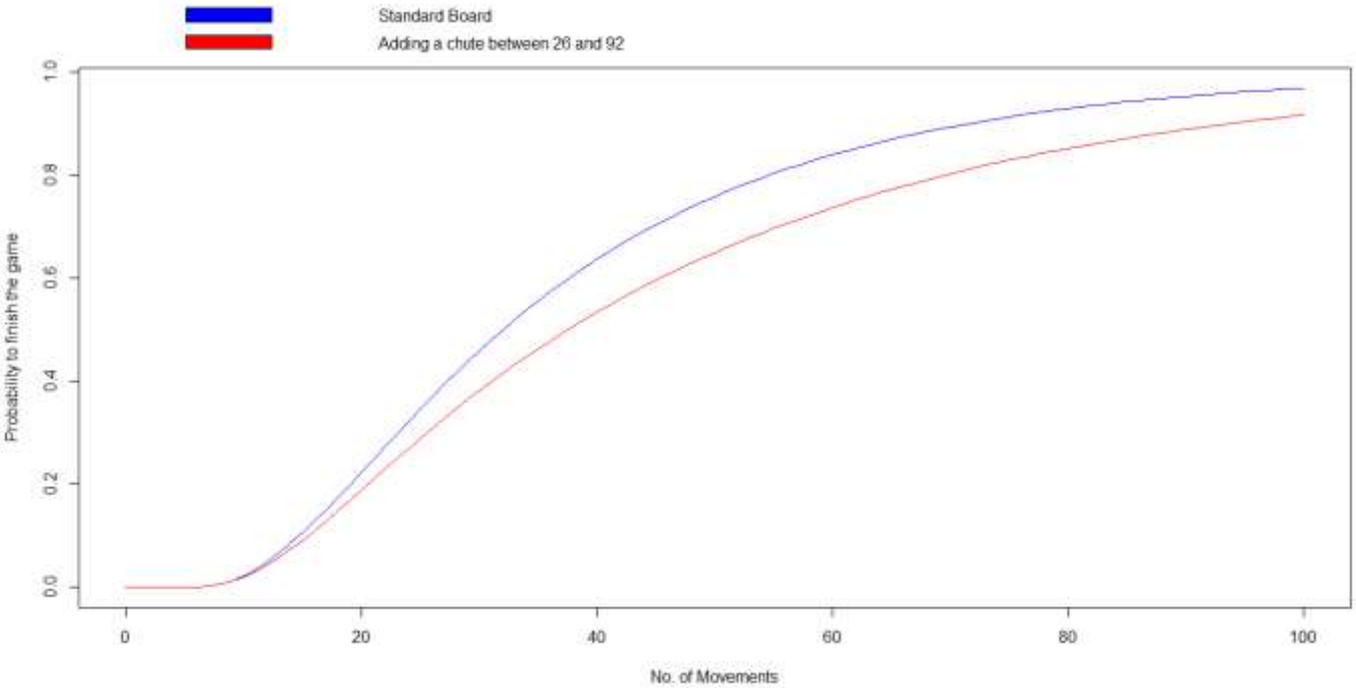

Figure 6

Clearly probability of finishing the game is decreased.

5. Conclusion

We introduce two new models , random staircase and Snakes and ladders models. We find the transition probabilities of snakes and ladders game and apply this result to the standard chutes and ladders model. The graph showing the probabilities to finish the game is an 'S' shaped curve. We have considered some special cases of the standard Chutes and ladder board also.

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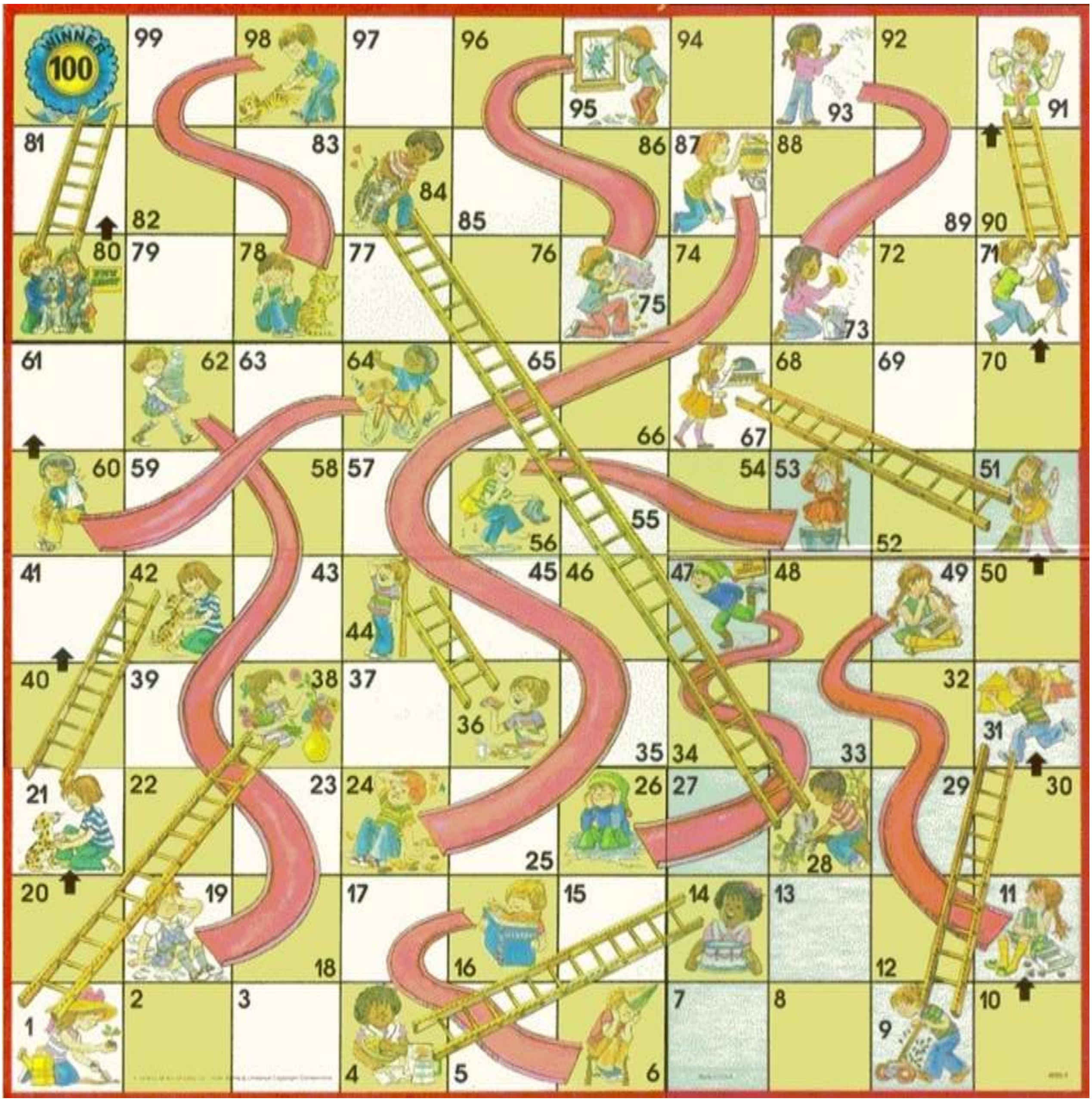
Appendix A

A.1 Snakes and ladders game rule[3]

Chutes and Ladders is played on a 100 square board game. There can be as many players as desired; however, since the actions of the players are independent from the actions of the other players, we shall only consider one player. The player begins off of the board, at a figurative square zero. The player then rolls a six-sided die, and advances the number of spaces shown on the die. For example, if the player is at position 8, and rolls a 5, they would advance to position 13. The game is finished when the player lands on square 100. There are two exceptions to the rule of movement. The first is that if the player, after advancing, lands on a chute (slide) or a ladder, they slide down or climb up them, respectively. For example, if the player, on their first turn, rolls a 4, the player advances to square 4, and then "climbs" to square 14, on the same turn. Thus, if the players is on square 77, and rolls a 3, the player is finished, as he advances to 80, and then climb to 100.The second is that the player must land exactly on square 100 to win. If the player rolls a die that would advance them beyond square 100, they stay at the same place. For example, a player at square 96 must roll exactly a 4 to win. A roll of 5 would make the player stay put at position 96. Do note that these rules could easily apply to a board of any size, with any size die, and with chutes and ladders in any position.

A.2 *Figure 7*



The Standard Chutes and Ladders Board[3]