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TOTAL BOUNDEDNESS OF A UNIFORM SPACE:-

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Abstract:-

This paper shows that in a uniform space $(\boldsymbol{X}, \boldsymbol{\mathcal{U}})$ every convergent filter is a Cauchy filter and every Cauchy filter converges to its accumulation points and in the context of uniform spaces, the uniformly continues image of a Cauchy filter is a Cauchy filter

Key-words:-

uniform space, filter, accumulation point, Totally Bounded, ultra Filter

Introduction:-

In topology and related branches of mathematics, a totally bounded space is a space that can be covered by finitely many subsets of every fixed "size" (where the meaning of "size" depends on the given context). The smallest the size fixed the more subsets may be needed but any specific size should require only finitely many subsets. A related nation is a totally bounded set in which only a subsets of the space needs to be covered. Every subsets of a totally bounded space is not totally bounded, some of its subsets still will be

A subset E of a Preudometric space (X, ρ) is called an $\in$- sized' subset if its

dim $E = \underset{x,y \in E}{\text{lub}} \rho(x,y) < \in$

If (x_n) is a sequence in X, then

$S_n = \{x_m : m \geq n\}; n \in N$

is the base for a filter F on X. In fact, given n, $k \in N$

let $m_0 = \min\{n,k\}$. Then $S_{m0} \subseteq S_n \cap S_k$.

If (x_n) is Cauchy-sequence in X for ρ and $\in > 0$, then $\dim S_m < \in$

for sufficiently large m and so the filter F contains an $\in$- sized set as a member for all $\in > 0$.

We now extend these notions to uniform spaces.

Definition:-

Let $(X, \mathcal{U})$ be a uniform space. A set $E \subseteq X$ is said to be U-sized

$U \in \mathcal{U}$ if $E \times E \subseteq U$.

By a u-cauchy filter in X we mean a filter F in X which contains a U-sized set as member for all $U \in \mathcal{U}$.

Proposition:-

In a uniform space $(X, \mathcal{U})$ every convergent filter is a cauchy -filter and every Cauchy filter converges to its accumulation points.

Proof:-

Suppose the filter $F \to x_0$ in $(X, \mathcal{U})$. Given $U \in \mathcal{U}$, we choose a symmetric $V \in \mathcal{U}$ such that $V \subseteq U$.

Since $F \to x_0$ for the nhd $V[x_0]$ of x_0, there is an element $A \in F$ such that $A \subseteq V[x_0]$. Hence $A \times A \subseteq V \subseteq U$. Thus the filter F contains a U-sized set A as a member for each $U \in \mathcal{U}$. Hence F is a Cauchy-filter on X. Assume next that F is a cauchy – filter on X and that $x_0 \in X$ is an accumulation point of F. For any $U \in \mathcal{U}$ let W be a closed symmetric vicinity such that $W \subseteq U$. Then for some $A_0 \in F$.

We have $\overline{A_0} \times \overline{A_0} = \overline{A_0 \times A_0} = \overline{W} = W$

Now since x_0 is an accumulation point of F

$x_0 \in \overline{A}$ for all $A \in F$. Hence

$x_0 \in \bigcap \left\{ \overline{A} : A \in F \right\} \subseteq \overline{A_0} \subseteq W[x_0] \subseteq U[x_0]$

Proposition:-

In the context of uniform spaces, the uniformly continuous image of a Cauchy filter is a Cauchy filter.

Proof:-

Let $f:(X,\mathcal{U})\to(Y,\vartheta)$ be a uniformly continuous map of a uniform space $(X,\mathcal{U})$ onto a uniform space (Y,ϑ). Suppose F is a

Cauchy filter in X. For each $V\in\vartheta$, there is a $U\in\mathcal{U}$ such that $(f(x),(y))\in V$ whenever x, y are two points of X with $(x,y)\in U$ since F is a Cauchy filter in X, F contains a U-sized set A as a member. Hence $(x,y)\in U$ for all $x,y\in A$. Thus $A\times A\subseteq U$. Thus for all $x,y\in A;(f(x),f(y))\in V$,

hence $f(A)\times f(A)\subseteq V$.

Thus $f(F)-\{f(A);A\in F\}$ contains a v-sized set $f(A)$ as a member. Thus $f(F)$ is a Cauchy-filter.

This completes the proof.

We next generalize the notion of uniform boundedness.

Let us recall that a pseudometric space (X,ρ) is said to be totally bounded (or precompact) if for each $\in>0$ there is a finite sequence of $\in$ -sized subsets which cover X, that is, iff X has a finite cover

$$\{S_c(x_i):1\le i\le n(\in)\}\text{ for all } \in>0$$

The notion of total boundedness can be generalized to uniform spaces.

Definition:-

A uniform space $(X,\mathcal{U})$ is said to be totally bounded if X has a finite U-sized cover for all $U\in\mathcal{U}$, that is, if for an arbitrary member $A\in\mathcal{U}$, there exist a finite member of members of X, say $x_1,.............,x_n$ such that

$$X=A[x_1]UA[x_2]U................ ......UA[x_n]$$

A subset E of a uniform space $(X,\mathcal{U})$ is said to be totally bounded in X whenever (E,U_E) is totally bounded.

Conclusion:-

Hence, In a uniform space $(X, \mathcal{U})$ every convergent filter is a cauchy -filter and every Cauchy filter converges to its accumulation points and In the context of uniform spaces, the uniformly continuous image of a Cauchy filter is a Cauchy filter

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