SOME PROPERTIES OF SEMI-PRECONNECTED SPACES

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Abstract: The aim of this paper is to introduce and discuss concepts of semi-preconnected spaces via the notion of semi-preopen sets as well as to investigate the characteristics and relationship to some well-known classes of topological spaces.

Index Terms: Semi-open set, preopen set, semi-continuity, precontinuity, semi-connected space, preconnected space.

1. Introduction:

The concept of semi-open set and semi-continuity was introduced by Norman Levin [3], P Das [2] defined the concept of semi connectedness in a topological space. The idea of preopen sets and precontinuity in topological spaces was introduced by Mashhour et al, [4]. Many mathematicians such as AP Balan Dhana, K Rao Chandrasekhar [1] have extended various concepts of preopen sets and preconnectedness in topological spaces. Here, we present some properties of semi preopen sets and semi-precontinuity in topological spaces.

2. Preliminaries:

All topological spaces considered in this paper lack separation axioms unless otherwise stated. We shall denote throughout this paper topological spaces by X and Y. A topological space consisting of the two points {0,1} with discrete topology will be denoted by $D_2$. We shall denote closure of A in X by cl(A) and interior of A by int(A).

2.1 Definition: ([3])

A subset A of topological space X is called semi-open iff there exists an open set $O \subset A \subset cl(O)$.

A set A in X is called semi-closed in X if its complement is semi-open in X.

2.2 Definition: ([3])

A subset A of topological space X is called preopen if $A \subset int(cl(A))$.

A set A in X is called preclosed in X if its complement is preopen in X.

2.3 Definition: ([1])

A subset A of a space X is said to be semi-preopen if there exists a preopen set U in X such that $U \subset A \subset cl(U)$.

2.4 Definition:

A topological space X is said to be connected if X cannot be expressed as union of two non-empty disjoint open sets.
2.5 Definition: ([4])

A function \( f : X \rightarrow Y \) is said to be semi-continuous (resp. pre-continuous) if inverse of every open set in \( Y \) is semi-open (resp. preopen) in \( X \).

3. Semi-Preconnected Spaces:

3.1 Definition: ([5])

A space \( X \) is called semi-preconnected if it is not union of two non-empty disjoint semi-open sets. A subset \( A \) of \( X \) is called semi-preconnected if it is semi-preconnected as a subspace of \( X \).

3.2 Theorem:

The following assertions are equivalent:

(i) Space \( X \) is semi-connected.
(ii) The only subsets of \( X \) both semi-preopen and semi-preclosed are \( \emptyset \) and \( X \).
(iii) No semi-precontinuous function \( f : X \rightarrow D_2 \) is surjective.

Proof: (i) \( \Rightarrow \) (ii)

\( G \subset X \) is both semi-preopen and semi-preclosed and \( G \neq \emptyset, X \); then \( X = G \cup G^c \).

Therefore, \( X \) is not semi-preconnected. By contrapositive, (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (iii)

If \( f : X \rightarrow D_2 \) were a semi-precontinuous surjective function,

then \( f^{-1}(\{0\}) \neq \emptyset, X \).

But \( \{0\} \) is both open and closed in \( D_2 \).

Hence, \( f^{-1}(\{0\}) \) is semi-preopen and semi-preclosed in \( X \).

This contradicts \( D_2 \).

(iii) \( \Rightarrow \) (i)

If \( X = A \cup B \), where \( A \) and \( B \) are disjoint non-empty semi-preopen sets,

then \( A \) and \( B \) are also semi-preclosed.

Let \( \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \) be characteristic function on \( X \).

Then, \( \chi_A(x) \) is a semi-precontinuous surjective function.

3.3 Theorem:

The semi-precontinuous image of a semi-preconnected space is semi-preconnected.

Proof: The map \( f : X \rightarrow f(X) \) is semi-precontinuous. If \( f(X) \) is not semi-preconnected, then there would be a semi-precontinuous surjection \( g : f(X) \rightarrow D_2 \). But then \( g \circ f : X \rightarrow D_2 \) would also be a semi-precontinuous surjective function, contradicting the semi-preconnectedness of \( X \).

3.4 Theorem:

Let \( Y \) be any space. The union of any family of semi-preconnected subsets having at least one point in common is also semi-preconnected.

Proof: Let \( C = \bigcup A_\alpha, y_0 \in \cap A_\alpha \) and \( f : C \rightarrow D_2 \) be semi-precontinuous.

Since each \( A_\alpha \) is semi-preconnected, no \( f|_{A_\alpha} \) is surjective.

Again, since \( y_0 \in A_\alpha \) for all \( \alpha \), \( f(y) = f(y_0) \) for all \( y \in A_\alpha \).
Hence, \( f \) cannot be surjective and so \( C \) is semi-preconnected.

### 3.5 Theorem:

Let \( f : X \to Y \) be continuous and open. If \( V \) be preopen in \( Y \) then \( f^{-1}(V) \) is also preopen in \( X \).

**Proof:** Let \( V \) be preopen set in \( Y \).

Then, there exists an open set \( W \) in \( Y \) such that

\[
V \subset W \subset cl(V).
\]

\[ \Rightarrow f^{-1}(V) \subset f^{-1}(W) \subset f^{-1}(cl(V)) \subset cl(f^{-1}(V)), \text{ as } f \text{ is open.} \]

But \( f^{-1}(W) \) is open as \( f \) is continuous.

Hence, \( f^{-1}(V) \) is preopen in \( X \).

### 3.6 Corollary:

Let \( f : X \to Y \) be continuous and open. If \( V \) be semi-preopen in \( Y \) then \( f^{-1}(V) \) is also semi-preopen in \( X \).

**Proof:** Let \( V \) be semi-preopen set in \( Y \).

Then, there exists preopen set \( U \) in \( Y \) such that

\[
U \subset V \subset cl(U).
\]

\[ \Rightarrow f^{-1}(U) \subset f^{-1}(V) \subset f^{-1}(cl(U)) \subset cl(f^{-1}(U)), \text{ as } f \text{ is continuous and open.} \]

But \( f^{-1}(U) \) is open as \( f \) is preopen, by theorem 3.5

Hence, \( f^{-1}(V) \) is semi-preopen in \( X \).

### 3.7 Theorem:

Let \( A \subset Y \) be semi-preconnected. Then any subset \( B \) satisfying \( A \subset B \subset semi\ pre\ cl(A) \) is also connected.

In particular, the semi-preclosure of a semi-preconnected set is semi-preconnected.

**Proof:** Let \( f : B \to D_2 \) be semi-precontinuous.

Then, since \( A \) is semi-preconnected, \( f|_A \) is not surjective.

But, \( B = semi\ pre\ cl(A) \cap B = semi\ pre\ cl_B(A) \)

Hence, \( f(B) = f(semi\ pre\ cl_B(A)) \)

\[ \subset pre\ cl_B(f(A)) \]

\[ \subset semi\ pre\ cl(f(A)), \text{ by precontinuity of } f \]

\[ = f(A), \text{ because } f(A) \text{ is preconnected.} \]

So, \( f \) cannot be surjective and \( B \) is semi-preconnected.

### 3.8 Theorem:

If \( f : X \to Y \) is open and precontinuous surjective function and \( X \) is semi-preconnected, then \( Y \) is connected.

**Proof:** Suppose \( Y \) is not connected. Then,

\[
Y = V_1 \cup V_2. \text{ Where } V_1 \cap V_2 = \emptyset \text{ and } V_1, V_2 \in \tau.
\]

So, \( X = f^{-1}(V_1) \cup f^{-1}(V_2) \).

\[ f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset. \]

Since \( V_i \) is both open and closed, \( V_i \) is preopen.

Since \( f \) is open and preopen, \( f^{-1}(V_i) \) is preopen, by theorem 3.5

Thus, \( X \) is not preconnected, a contradiction. Hence, \( Y \) must be connected.
3.9 Theorem:

Let A be preconnected subspace of X. If B be a subspace of X such that

\[ A \subset B \subset \text{cl}(A) \]

then B is preconnected.

**Proof:** Let \( B = G \cup H \), a disconnection by preclosed sets G and H in X.

Then, \( G = B \cap E, H = B \cap F \) where E and F are preclosed in X.

Since A is preconnected, \( A \subset G \) or \( A \subset H \).

Suppose \( A \subset G = B \cap E \subset E \). Then, \( \text{cl}(A) \subset \text{cl}(E) = E \).

Hence, \( B \subset E \).

Also, \( B \subset B \cap E = G \Rightarrow G \cup H \subset G \), which is impossible.

Therefore, B is preconnected.

4. Conclusion:

In this paper, we see that properties similar to semi-open sets and semi-continuity can also be derived for semi-preopen set and semi-precontinuity in topological spaces.

5. References:


