STUDIES OF OBSERVATIONAL RELATIONS FOR COSMOLOGICAL MODELS

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Abstract
In this article we discuss observations in a general curved space-time. Although the sources and observer move with a unique velocity at each space-time point in the applications to cosmological models we have in mind, many of the relations are valid for sources and observers moving with arbitrary 4-velocities at arbitrary points. We have derived observational relations valid for any cosmological model and have given an illustration of the methods using their application to the standard isotropic cosmological models.

Key Words
Cosmological models, geometric, red shifts, symmetry, geodesics and dominant energy

The geometric approximation
The radiation which conveys information in a cosmological model is represented by geometric optics solution of Maxwell’s equation. This electromagnetic field $F_{ab}$ is regarded as a test-field (i.e., we can neglect its effect on the curvature of space-time) in a charge and current free space-time, and so obey Maxwell’s source-free equation.

\[ F_{ab;c} = 0 \Leftrightarrow \exists \Phi_a : F_{ab} = \Phi_{b,a} - \Phi_{a;b} \]  
\[ F_{a;b} = 0. \]

The potential will be chosen to obey the gauge condition

\[ \Phi_{a,a} = 0. \]  

We assume there exist solution of these equation of the form

\[ \Phi_a = g(\Psi) A_a + \text{small terms}, \]  

Where $g(\Psi)$ is an arbitrary function of the phase $\Psi$ and $g$ varies rapidly compared with the amplitude $A_a$ in the sense that

\[ g^\ast a A_b >> g A_{a;b}. \]  

Where $g^\ast = \frac{\delta g}{\delta \Psi}$ and we have defined the propagation vector $k_a$ by

\[ k_a = \Psi_{;a}. \]  

Substituting (4) into (2), ignoring the tail terms, and equating to zero separately the coefficients of $g$, $g'$ and $g''$ (which we may do as $g$ is arbitrary), we find

\[ k^a k_a = 0, \]  
\[ A_{a;b} k^b = 0 - \frac{1}{2} A_{a;a} k^b, \]  
\[ (A^a_{;b})_{;b} + R^c_{a;b} A^b = 0. \]
The third equation will play no further part in the present discussion, it is essential effect shows that we cannot in
general omit tail terms if (4) is to be an exact solution of the geometric approximation.

Form (1) and (5) we find that the electro-magnetic field has the approximate form

$$F_{ab} = g'(K_a A_b - A_a K_b)$$  \hspace{1cm} (10)

and the electromagnetic stress tensor $S_{ab}$, defined by

$$S_{ab} = F_{ab}F^b - \frac{1}{2} g_{ab} F^{cd} F_{cd},$$

has the form

$$S_{ab} = A^2 (g')^2 k_a k_b,$$  \hspace{1cm} (11)

where we have defined

$$A^2 = A^a A^a.$$

An observer with 4-velocity $u^a$ finds the radiation flux across a surface perpendicular to $k^a$ to be the same as the
instantaneous energy density of the radiation,

$$S_{ab} u^a u^b = A^2 (g')^2 (k_a u^a)^2.$$  \hspace{1cm} (12)

Equation (7) implies $k^a k_{a;b} = 0$. However (6) shows $k_{a;b} = k_{b;a}$, so we find

$$k_a k_{a;b} = 0.$$  \hspace{1cm} (13)

Thus the light rays (the curves whose tangent vector field is $k$) are null geodesics. It follows that light rays are
bent by an anisotropic gravitational field.

**Red shifts**

The rate of change of $g(\nu)$ measured by an observer moving with 4-velocity $u^a$ is

$$\frac{d}{dt} g(\nu) = g(k^a u_a).$$

If observers with 4-velocities measure the rate of change of the same signal $g(\nu)$, these rates of change are in the
ratio

$$\frac{(k^a u_a)^1}{(k^a u_a)^2} = \frac{\nu_1}{\nu_2}. $$

We can think of this as a time-dilatation effect: if a proper time interval $dt$ is observed to elapse between particular signals, then

$$\frac{d}{dt} = \frac{(k^a u_a)^1}{(k^a u_a)^2}. $$

In particular, the observed frequencies of light or radio waves are related by

$$\frac{\nu_1}{\nu_2} = \frac{(k^a u_a)^1}{(k^a u_a)^2}. $$

The red-shift $z$ of a source as measured by an observer is defined in terms of wave lengths by

$$z = \frac{\lambda_{observed} - \lambda_{emitted}}{\lambda_{emitted}} = \frac{\Delta \lambda}{\lambda_{emitted}}.$$  \hspace{1cm} (14)

We therefore find that

$$1 + z = \frac{\lambda_{observed}}{\lambda_{emitted}} = \frac{\nu_{emitted}}{\nu_{observed}}.$$  \hspace{1cm} (15)

Determined the red-shift from the 4-velocity vectors $(u^a)$ observer, $(u^a)_{emitter}$ and from the tangent vector $k^a$ to the
null geodesic. This relation is true no matter what the separation of emitter and observer, and holds independent of
the interpretation of the red-shift as a Doppler or gravitational red-shift. As an example, if both observer and emitter are at the same point and the emitter moves radically away from the observer, we can write

\[ u_a = \cosh \beta \ u_a^* + \sin \beta e_a^* \ e_a^* e_a^* = 1, \ e_a^* e_a^* = 0. \]  

(16)

as the 4-velocity of the emitter, where \( u_a^* \) is the observer’s 4-velocity, \( e_a \) is the direction of motion of the emitter in the observer’s rest frame, and \( V = \tan \beta \) is the velocity of relative motion. Then a null ray representing a signal from the emitter to the observer is

\[ k^a = k \left( u^a - e^a \right); \]

from (15) we immediately find

\[ 1 + z = \exp (-\beta) = \sqrt{\frac{1 + V}{1 - V}}, \]

(17)

the standard result for the red-shift due to radial motion of source relative to the observer when both are at the same space-time point.

For later use, we introduce the following decomposition of \( k^a \). We consider an observer with 4-velocity \( u_a \) and let \( v \) be an affine parameter along the null geodesic with tangent vector \( k_a \); If \( n_a \) is the unit vector in the direction of the projection of \( k_a \) into the rest space of the observer, then

\[ k^a = \left( -u_a, k^b \right) \left( u^a + n^a \right), \ n^a n_a = 1, \ n^a u_a = 0. \]  

(18)

Thus a small increment \( dv \) in the affine parameter will be considered by the observer to correspond to a time difference \( dt \) and a spatial displacement \( dl \), with

\[ |dt| = |dv| = (-k^a u_a) \ dv. \]  

(19)

The linear red-shift relation in a cosmological model

In a given cosmological model, the emitter and observer coincide with particular galaxies moving with the unique fluid velocity \( u^a \). The change in \( (u^a k_a) \) occurring in a parameter distance \( dv \) along the null geodesic is

\[ d \left( u^a k_a \right) = \left( u^a k_a \right) k^a dv \]

\[ = \left( -u_a k^b \right) dv + u_a \left( k^a k^b \right) dv. \]

The second term vanishes by (13). Substituting from (14) and (18),

\[ d \left( u^a k_a \right) = \left( \theta_a n^a n^b + u_a n^a \right) \left( u^a k_a \right) ^2 dv. \]

As (13) implies that the change \( d \) in any wave length in the parameter distance \( dv \) is given by

\[ \frac{d \lambda}{\lambda} = -\frac{d \left( u^a k_a \right)}{u^a k_a}, \]

the change of red-shift along the null geodesic is

\[ \frac{d \lambda}{\lambda} = \left( \theta_a n^a n^b + u_a n^a \right) dt. \]  

(20)

Using (16), this can be written

\[ \frac{d \lambda}{\lambda} = \left( dt \right) \left( u^a n_a \right) dt. \]  

(21)

Thus the red-shift has been split (because we have a unique 4-velocity \( u^a \) determined at each point) into a radical Doppler part and a gravitational part (the second term). Further, we can see how this red-shift-distance relation varies with direction in the sky; since the angular dependence of the term due to \( q_a \) and \( u_a^* \) are different, we can in principle determine these quantities directly from the linear red-shift-distance relation around that point.

Spherically symmetric cosmological models

To illustrate the above relations, we consider a Robertson-Walker universe in its coordinates. By the homogeneity and isotropy of this space-time, all further-directed null geodesics are equivalent, so it suffices to consider future-directed radial null geodesics through the origin of coordinates. The corresponding solution of the geodesic equation is
\[
k^a = \frac{1}{R} \left( 1, \frac{1}{R}, 0, 0 \right) \Rightarrow k_0 = \frac{1}{R} \left( -1, R, 0, 0 \right).
\]

Since the four velocity vector is \( u^a = (1, 0, 0, 0) \), we find
\[
- u^a k_a = \frac{1}{R}
\]
Thus in these models,
\[
1 + z = \frac{R_{\text{observers}}}{R_{\text{emitter}}}
\]
Further, a small affine parameter change \( dv \) along the null geodesics corresponds to coordinate changes \( dt, dr \)
where
\[
dv = Rd\tau = R^2dr,
\]
from eq. (19)
The divergence of \( k^a \) is
\[
k^a = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta x^a} \left( \sqrt{-g} k^2 \right)
\]
As \( r \) is the radial co-ordinates difference between the source and the observer, we note that
Where \( t_0 \) is the time at the observer and \( t_G \) the time at the source.
Thus if we define
\[
u = \int_{t_0}^{t} \frac{dt}{R(t)}
\]
then we have
\[
r_G^2 = [R^2]_o f^2(u),
\]
where
\[
f(u) = \sin u, \text{ if } k = +1
\]
\[
= u, \text{ if } k = 0
\]
\[
= \sin hu, \text{ if } k = -1
\]
Similarly, we can show that
\[
r_o^2 = [R^2]_o f^2(u),
\]
\[
= (1 + z)^2 [R^2] f^2(u),
\]
Which is consistent with (23).
Whenever we can express \( u \) in the form
\[
u = \int_{R_0}^{R} \frac{dR}{R} = \int_{z}^{z_o} \frac{dz}{(1 + z)R}
\]
and then substitute from the Friedmann equation to obtain \( u(z) \) and so \( r_0(z) \). for example, when pressure free matter is the dominant energy component in the universe, we can use the Friedmann equation in the form
\[
h^2 = \left[ \frac{f_o}{t} \right]^2 = \left( \frac{f_o}{t} \right)^2 \left( 2 q_o H_o^2 + \frac{3}{2} \right) + \frac{\hat{\omega}}{2} - \left( \frac{f_o}{t} \right)^2 K(t_0)
\]
and numerically integrate to obtain \( r_0(z) \). In the particular case , we can integrate analytically to find that
\[
r_o = \frac{1}{H_o q_o^2 (1 + z)} \left[ q_o \frac{z}{z} + (q_o - 1) \left( (1 + 2 q_o z)^2 - 1 \right) \right],
\]
when and
\[
r_o = \frac{1}{2 H_o \left( 1 - \frac{1}{(1 + z)^2} \right)},
\]
when \( q_o = 0 \).
We, thus, see that when, there is a maximum value of \( r_0 \) for some value of \( z \), thereafter, \( r_0 \) decreases as \( z \) increases; i.e., we do have refocusing of null geodesics in these models. This is still true if we include a pressure term in the Friedmann equation, so it is true in all non-empty Robertson-Walker universes which expand from a singularity (we can always ignore a term at early enough times).

If we take the matter content of the universe to be dust and ignore the effects of radiation on the dynamics of the universe, we get

\[
(k_u u^a) dv = -dt
\]

(31)

\[
= -\frac{1}{R} \frac{dR}{dz} \frac{dz}{(1 + z)^2 \left( \frac{3}{2} + \frac{2q_u z}{1 + q_u z} \right) \frac{1}{(1 + z)^2}}
\]

For simplicity we shall further take and assume that the radiation sources and absorbing particles are conserved; then

\[
n(z) = n(O) \left( \frac{R_O^3}{R^3} \right) = n(O) (1 + z)^3
\]

Hence the contribution to intensity from sources up to a red-shift \( z^* \) is given

\[
I_\nu = \frac{n(O)}{H_\nu} \int_{z^*}^\infty \frac{s(z, \nu) (1 + z)}{(1 + z)^2 \left( \frac{3}{2} + \frac{2q_u z}{1 + q_u z} \right) \frac{1}{(1 + z)^2}} d\nu
\]

(32)

We might further assume that the source emission can be described by a source spectrum \( \Phi(\nu) \) which is independent of \( z \), and an amplitude \( S(z) \); then

\[
G_\nu(\nu_0) = \frac{n(O)}{H_\nu} \frac{S(z)}{(1 + z)^2 \left( \frac{3}{2} + \frac{2q_u z}{1 + q_u z} \right) \frac{1}{(1 + z)^2}}
\]

(33)

If the frequency be \( \nu_0 \), we can put

\[
G_\nu(\nu_0) = 0, \quad \text{if} \quad z > z^*
\]

and then we can rewrite (6.24) in the form

\[
I_\nu = \int_{z^*}^\infty \Phi(\nu_0) G_\nu(\nu_0) d\nu_0
\]

(34)

If the sources have spectral index \( q_0 \) and their amplitude varies as \( (R(t))^n \), then

\[
S(t, \nu (1 + z)) = S(t, \nu_0) (1 + z)^n
\]

so in an Einstein-de Sitter universe \( q_0 = \frac{1}{2} \), we find form (32) that

\[
I_\nu = \frac{n(O) S(\nu_0, t_\nu)}{H_\nu \left( \frac{2}{3} + \frac{m}{2} \right) \left( 1 + z_\nu \right)^{\frac{3}{2} + \frac{2q_u z}{1 + q_u z}}}
\]

(35)

when \( m = \frac{\infty + \frac{3}{2}}{2} \)

and in a Milne universe \( (q_0 = 0) \), we find

\[
I_\nu = \frac{n(O) S(\nu_0, t_\nu)}{H_\nu \left( \frac{2}{3} + \frac{m}{2} \right) \left( 1 + z_\nu \right)^{\frac{3}{2} + \frac{2q_u z}{1 + q_u z}}}
\]

(36)

when \( m = \frac{\infty + 1}{2} \).

In the exceptional cases

\[
q_0 = \frac{1}{2}, \quad m = \frac{\infty + \frac{3}{2}}{2}
\]

and \( q_0 = 0, \quad m = \frac{\infty + 1}{2} \)

we find

\[
I_\nu = \frac{n(O) S(\nu_0, t_\nu)}{H_\nu} \log (1 + z_\nu).
\]
Thus if the source brightness increases faster than \((1 + z)^{-1}\) in a Milne universe, there must be a cut-off in the source evolution before some value of z which can be determined by comparing these expression, with the observed values of \(10^\nu\). The source of radiation might be galaxies or other discrete sources.

The optical depth up to red-shift z in a Robertson-Walker universe is given by

\[
p(z, \nu) = \int_z^\infty \frac{n(\nu)}{H_a} \sigma(z', \nu (1 + z')) \frac{(1 + z')}{(1 + 2 q_o z')^2} \, dz'
\]  

(35)

Considering some particular scattering or absorption process, it is reasonable to assume

\[
\sigma(z, \nu) = \sigma(z) A(\nu).
\]  

(36)

In particular, we can represent time absorption at some frequency \(\nu\), setting \(A(\nu, \nu) = \delta(\nu - \nu_o)\); the resulting optical depth \(A(\nu, \nu)\) is given by

\[
A(\nu, \nu_o) = \frac{n(\nu)}{H_a \nu_o} \sigma(z) \frac{(1 + z)^2}{(1 + 2 q_o z)^2} \text{ for } z < z_o = 0,
\]

for \(A \geq z_o\).

where \(z_o = (\nu_o / \nu) - 1\)

Regarding an absorption process with cross section, we can re express (34) in the form

\[
p(z, \nu_o) = \int_z^\infty A(\nu, \nu_o) A(\nu_o) \, d\nu_o.
\]  

(37)

While in general we have to integrate these equation numerically, in special cases we can integrated them analytically. In particular, when \(\sigma = (z, \nu) = 0\) \(\sigma_o = \text{const}\), we can integrate (34) and get

\[
p(z, \nu) = \frac{\sigma(n_o)}{H_a} \frac{1}{3 q_o^2} \left( (3 q_o + 3 q_o z - 1) \left( 1 + 2 q_o z \right)^{\frac{1}{2}} - (3 q_o + 1) \right),
\]

when \(q_o > 0\) and

\[
p(z, \nu) = \frac{\sigma(n_o)}{H_a} \frac{1}{2} (1 + z)^{\frac{1}{2}},
\]

when \(q_o = 0\).

Conclusion

Finally, we may remark that while we have been considering the above equations in the context of a cosmological model, one can apply them in other situations when a fluid approximation is appropriate.

References

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