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STUDIES OF OBSERVATIONAL RELATIONS FOR COSMOLOGICAL MODELS

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Abstract

In this article we discuss observations in a general curved space-time. Although the sources and observer move with a unique velocity at each space-time point in the applications to cosmological models we have in mind, many of the relations are valid for sources and observers moving with arbitrary 4-velocities at arbitrary points. We have derived observational relations valid for any cosmological model and have given an illustration of the methods using their application to the standard isotropic cosmological models.

Key Words

Cosmological models, geometric, red shifts, symmetry, geodesics and dominant energy

The geometric approximation

The radiation which conveys information in a cosmological model is represented by geometric optics solution of Maxwell's equation. This electromagnetic field F_{ab} is regarded as a test-field (i.e., we can neglect its effect on the curvature of space-time) in a charge and current free space-time, and so obey Maxwell's source-free equation.

$$F_{ab;c} = 0 \Leftrightarrow \exists \Phi_a : F_{ab} = \Phi_{b;a} - \Phi_{a;b}, \qquad (1)$$

$$F^{ab}_{;b} = 0, \qquad (2)$$

The potential will be chosen to obey the gauge condition

$$\Phi^a_{;a} = 0. \qquad (3)$$

We assume there exist solution of these equation of the form

$$\Phi_a = g(\Psi) A_a + \text{smal terms}, \qquad (4)$$

Where $g(\Psi)$ is an arbitrary function of the phase Ψ and g varies rapidly compared with the amplitude A_a in the sense that

$$g'k_a A_b >> g A_{a;b} \qquad (5)$$

Where $g' = \frac{\delta g}{\delta \Psi}$ and we have defined the propagation vector k_a by

$$k_a = \Psi_{;a}. \qquad (6)$$

Substituting (4) into (2), ignoring the tail terms, and equating to zero separately the coefficients of g, g' and g" (which we may do as g is arbitrary), we find

$$k^a k_a = 0, \qquad (7)$$

$$A_{a;b} k^b = 0 \frac{1}{2} A_a k^b_{;b}, \qquad (8)$$

$$(A^a)^{;b}_{;b} + R^a_b A^b = 0. \qquad (9)$$

The third equation will play no further part in the present discussion, it is essential effect shows that we cannot in general omit tail terms if (4) is to be an exact solution of the geometric approximation.

Form (1) and (5) we find that the electro-magnetic field has the approximate form

$$F_{ab} \approx g'(K_a A_b - A_a K_b) \qquad (10)$$

and the electromagnetic stress tensor S_{ab}, defined by

$$S_{ab} = F_{ab} F_b^c - \frac{1}{2} g_{ab} F^{cd} Fcd,$$

has the form

$$S_{ab} \approx A^2 (g')^2 k_a k_b, \qquad (11)$$

where we have defined

$$A^2 = A^a A_a.$$

An observer with 4-velocity u^a finds the radiation flux across a surface perpendicular to k^a to be the same as the instantaneous energy density of the radiation,

$$S_{ab} u^a u^b = A^2 (g')^2 (k_a u^a)^2. \qquad (12)$$

Equation (7) implies $k^a k_{a;b} = 0$. However (6) shows $k_{a;b} = k_{b;a}$, so we find

$$k_{a;b} k^b = 0. \qquad (13)$$

Thus the light rays (the curves whose tangent vector field is k^a) are null geodesics. It follows that light rays are bent by an anisotropic gravitational field.

Red shifts

The rate of change of $g(\Psi)$ measured by an observer moving with 4-velocity u^a is

$$g_{;a} u^a = g'(k_a u^a).$$

If observers with 4-velocities measure the rate of change of the same signal $g(\Psi)$, these rates of change are in the ratio

$$\frac{(k_a u^a)_1}{(k_b u^b)_2}$$

We can think of this as a time-dilatation effect: if a proper time interval dt is observed to elapse between particular signals, then

$$\frac{dt_2}{dt_1} = \frac{(k_a u^a)_1}{(k_b u^b)_2}.$$

In particular, the observed frequencies of light or radio waves are related by

$$\frac{\upsilon_1}{\upsilon_2} = \frac{(k_a u^a)_1}{(k_b u^b)_2}$$

The red-shift z of a source as measured by an observer is defined in terms of wave lengths by

$$z = \frac{\lambda\ observed - \lambda\ emitted}{\lambda\ emitted} = \frac{\Delta\lambda}{\lambda\ emitted}. \qquad (14)$$

We therefore find that

$$1 + z = \frac{\lambda\ observed}{\lambda\ emitted} = \frac{\upsilon\ emitted}{\upsilon\ observed},$$

So

$$1 + z = \frac{(u^a k_a)\ emitter}{(u^b k_b)\ observer} \qquad (15)$$

Determines the red-shift from the 4-velocity vectors (u^a) observer, $(u^a)_{emitter}$ and from the tangent vector k^a to the null geodesic. This relation is true no matter what the separation of emitter and observer, and holds independent of the interpretation of the red-shift as a Doppler or gravitational red-shift. As an example, if both observer and emitter are at the same point and the emitter moves radically away from the observer, we can write

$$u_a = \cosh\beta\, u_a + \sin h\,\beta e_a,\ e_a e^a = 1,\ e^a e_a = 0, \quad (16)$$

as the 4-velocity of the emitter, where u^a is the observer's 4-velocity, e_a is the direction of motion of the emitter in the observer's rest frame, and V = tan hβ is the velocity of relative motion. Then a null ray representing a signal from the emitter to the observer is

$$k^a = k\left(u^a - e^a\right);$$

from (15) we immediately find

$$1 + z = \exp(-\beta) = \sqrt{\frac{1+v}{1-V}}, \quad (17)$$

the standard result for the red-shift due to radial motion of source relative to the observer when both are at the same space-time point.

For later use, we introduce the following decomposition of k^a. We consider an observer with 4-velocity u^a and let v be an affine parameter along the null geodesic with tangent vector k^a;

$$k^a = \frac{dx^a}{dv}$$

If n^a is the unit vector in the direction of the projection of k^a into the rest space of the observer, then

$$k^a = \left(-u_b k^b\right)\left(u^a + n^a\right),\ n^a n_a = 1,\ n^a u_a = 0. \quad (18)$$

Thus a small increment dv in the affine parameter will be considered by the observer to correspond to a time difference dt and a spatial displacment d*l*, with

$$|dt| = |dl| = \left(-k^a u_a\right)dv. \quad (19)$$

The linear red-shift relation in a cosmological model

In a given cosmological model, the emitter and observer coincide with particular galaxies moving with the unique fluid velocity u^a. The change in ($u^a k_a$) occurring in a parameter distance dv along the null geodesic is

$$d\left(u^a k_a\right) = \left(u^a k_a\right)_{;b} k^v dv$$

$$= \left(u_{a;b} k^a k^b\right)dv + u_a\left(k^a_{;b} k^b\right)dv.$$

The second term vanishes by (13). Substituting from (14) and (18),

$$d\left(u^a k_a\right) = \left(\theta_{ab} n^a n^b + \dot{u}_a n^a\right)\left(u^c k_c\right)^2 dv.$$

As (13) implies that the change d in any wave length in the parameter distance dv is given by

$$\frac{d\lambda}{\lambda} = -\frac{d\left(u^a k_a\right)}{\left(u^b k_b\right)},$$

the change of red-shift along the null geodesic is

$$\frac{d\lambda}{\lambda} = \left(\theta_{ab} n^a n^b + \dot{u}^a n^a\right)dl. \quad (20)$$

Using (16), this can be written

$$\frac{d\lambda}{\lambda} = (dl)^{\cdot}\left(\dot{u}^a n_a\right)dl. \quad (21)$$

Thus the red-shift has been spilt (because we have a unique 4-velocity u^a determined at each point) into a radical Doppler part and a gravitational part (the second term). Further, we can see how this red-shift-distance relation varies with direction in the sky; since the angular dependence of the term due to q, σ_{ab} and $\dot{u}^a$ are different, we can in principle determine these quantities directly from the linear red-shift-distance relation around that point.

Spherically symmetric cosmological models

To illustrate the above relations, we consider a Robertson-Walker universe in its coordinates. By the homogeneity and isotropy of this space-time, all further-directed null geodesics are equivalent, so it suffices to consider future-directed radial null geodesics through the origin of coordinates. The corresponding solution of the geodesic equation is

$$k^a = \frac{1}{R}\left(1, \frac{1}{R}, O, O\right) \Leftrightarrow k_a = \frac{1}{R}(-1, R, O, O). \quad (22)$$

Since the four velocity vector is $u^a = (1, 0, 0\ 0)$, we find

$$-u^a k_a = \frac{1}{R} \quad (23)$$

Thus in these models,

$$1 + z = \frac{R_{observer}}{R_{emitter}}. \quad (24)$$

Further, a small affine parameter changes dv along the null geodesics corresponds to coordinate changes dt, dr where

$$dv = Rdt = R^2\, dr, \quad (25)$$

from eq. (19)

The divergence of k^a is

$$k^a{}_{;a} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta x^2}\left(\sqrt{-g}\, k^2\right)$$

$$= \frac{2}{R^2}\left[\dot{R} + \frac{f'(r)}{f(r)}\right].$$

As r is the radial co-ordinates difference between the source and the observer, we note that

$$r = \int_G^o dr = \int_{tG}^{to} \frac{dt}{R(t)},$$

Where t_0 is the time at the observer and t_G the time at the source.

Thus if we define

$$u = \int_{tG}^{to} \frac{dt}{R(t)}, \quad (26)$$

then we have

$$r_G^2 = [R^2]_o f^2(u), \quad (27)$$

where

$f(u) = \sin u$, if $k = +1$

$= u$, if $k = 0$

$= \sin hu$, if $k = -1$

Similarly, we can show that

$$r_o^2 = [R^2]_G\ f^2(u), \quad (28)$$

$$= (1 + z)^{-2}\, [R^2]\ f^2(u),$$

Which is consistent with (23).

Whenever we can express u in the form

$$u = \int_{RG}^{Ro} \frac{dR}{R\dot{R}} = \int_o^z \frac{dz}{(1+z)\dot{R}},$$

and then substitute from the Friedmann equation to obtain u(z) and so $r_0(z)$. for example, when pressure free matter is the dominant energy component in the universe, we can use the Friedmann equation in the form

$$H^2 = \left(\frac{\dot{\iota}}{\iota}\right)^2 = \left(\frac{\iota_o}{\iota}\right)\left(2_{qo} H_0^2 + \frac{3}{2}\wedge\right) + \frac{\wedge}{2} - \left(\frac{\iota_o}{\iota}\right)^2 K(t_0)$$

and numerically integrate to obtain $r_0(z)$. In the particular case , we can integrate analytically to find that

$$r_o = \frac{1}{H_o q_o^2 (1+z)^2}[q o^z + (q_o - 1)\{(1 + 2q_o z)^{\frac{1}{2}} - 1\}], \quad (29)$$

when and

$$r_o = \frac{1}{2H_o}\left(1 - \frac{1}{(1+z)^2}\right), \quad (30)$$

when $q_0 = 0$.

We, thus, see that when , there is a maximum value of r_0 for some value of z, thereafter, r_0 decreases as z increases; i.e., we do have refocusing of null geodesics in these models. This is still true if we include a pressure term in the Friedmann equation, so it is true in all non-empty Robertson-Walker universes which expand from a singularity (we can always ignore a term at early enough times).

If we take the matter content of the universe to be dust and ignore the effects of radiation on the dynamics of the universe, we get

$$(k_a u^a)\, dv = -dt \qquad (31)$$

$$= -\frac{1}{R}\frac{dR}{dz}dz$$

$$= \frac{dz}{(1+z)^2\{H_o^2(1+2q_o z) + (\wedge/3)\left(2z - 1 + (1+z)^{-2}\right)\}^{\frac{1}{2}}}$$

For simplicity we shall further take and assume that the radiation sources and absorbing particles are conserved; then

$$n(z) = n(O)\,(R_O^3/R^3)$$

$$= n(O)\,(1+z)^3$$

Hence the contribution to intensity from sources up to a red-shift Z_* is given

$$I\upsilon = \frac{n(O)}{H_o}\int_o^{z_*}\frac{s(z, \upsilon(1+z))}{(1+z)^2\,(1+2q_o z)^{\frac{1}{2}}} \qquad (32)$$

We might further assume that the source emission can be described by a source spectrum $\Phi(\upsilon)$ which is independent of z, and an amplitude $\bar{S}(z)$; then

$$S(z,\upsilon) = \bar{S}(z)\,\Phi(\upsilon). \qquad (33)$$

If the frequency be υ_e, we can put

$$\Phi(\upsilon) = \upsilon(\upsilon - \upsilon_e)$$

and get the corresponding emission $G_\upsilon(\upsilon_e)$

$$G_\upsilon(\upsilon_e) = \frac{n(O)}{H_o\upsilon_e}\frac{\bar{S}(z)}{(1+z\upsilon)(1+2q_o z_\upsilon)^{\frac{1}{2}}} \text{ if } z_v < z_*$$

$$= O, \quad \text{if } z_\upsilon > z^*$$

where $z_\upsilon = (\upsilon_e/\upsilon) - 1$

and then we can rewrite (6.24) in the form

$$I_\upsilon = \int_o^\infty \Phi(\upsilon_e)\,G_\upsilon(\upsilon_e)\,d\upsilon_e. \qquad (34)$$

If the sources have spectral index $\propto$ and their amplitude varies as $(R(t)^m$, then

$$S(t, \upsilon(1+z) = S(t_o, \upsilon)(1+z)^{m-\propto}$$

so in an Einstein-de Sitter universe $\left(q_o = \frac{1}{2}\right)$ we find form (32) that

$$I\upsilon = \frac{n(O)S(\upsilon, t_o)}{H_o\left(\propto + \dfrac{2}{3} - m\right)}\{1 - \frac{1}{(1+z_*)^{3/2+\propto-m}}\},$$

when $m \neq \propto + \frac{3}{2}$.

and in a Milne universe ($q_O = 0$) , we find

$$I\upsilon = \frac{n(O)S(\upsilon, t_o)}{H_o(\propto + 1 - m)}\{1 - \frac{1}{(1+z_*)^{1+\propto-m}}\}$$

when $m \neq \propto + 1.$

In the exceptional cases

$$q_o = \frac{1}{2}, \quad m = \propto + \frac{3}{2}$$

and $\quad q_o = 0, \quad m = \propto + 1$

we find

$$I\upsilon = \frac{n(O)S(\upsilon, t_o)}{H_o}\log(1+z_*).$$

Thus if the source brightness increases faster than $(1+z)^{\infty+1}$ in a Milne universe, there must be a cut-off in the source evolution before some value of z which can be determined by comparing these expression, with the observed values of $I\upsilon$. The source of radiation might be galaxies or other discrete sources.

The optical depth up to red-shift z in a Robertson-Walker universe is given by

$$p(z, v) = \int_o^z \frac{n(o)}{H_o} \sigma(z', \upsilon(1+z')) \frac{(1+z')dz'}{(1+2q_o z')^{\frac{1}{2}}} \qquad (35)$$

Considering some particular scattering or absorption process, it is reasonable to assume

$$\sigma(z,\upsilon) = \overline{\sigma}(z)\, A(\upsilon). \qquad (36)$$

In particular, we can represent time absorption at some frequency υ_a setting $A_\upsilon(z, \upsilon_\upsilon) = \delta(\upsilon - \upsilon_a)$; the resulting optical depth $A_\upsilon(z, \upsilon_\upsilon)$ is given by

$$A_\upsilon(z, \upsilon_a) = \frac{n_a^{(0)}}{H_o \upsilon_a} \overline{\sigma}(z\upsilon) \frac{(1+z)^2}{(1+2q_o z\upsilon)^{\frac{1}{2}}}, \text{ for } z\upsilon < z_*$$

$$= 0, \qquad \text{for } A\ z_\upsilon \geq z_*$$

where $z_\upsilon = (\upsilon_a / \upsilon) - 1$

Regarding an absorption process with cross section, we can re express (34) in the form

$$p(z, \upsilon_a) = \int_o^\infty A_\upsilon(z, \upsilon_a)\, A(\upsilon_a)\, d\upsilon_a. \qquad (37)$$

While in general we have to integrate these equation numerically, in special cases we can integrated them analytically. In particular, when $\sigma = (z,\upsilon) = 0 \quad \sigma_o$ = const, we can integrate (34) and get

$$p(z, \upsilon) = \frac{\sigma_o n_a^{(o)}}{H_o} \cdot \frac{1}{3q_o^2} \{(3q_o + 3_o z - 1)(1 + 2_q z)^{\frac{1}{2}} - (3q_o + -1)\},$$

when $q_o \neq 0$ and

$$p(z, \upsilon) = \frac{\sigma_o n_a^{(o)}}{H_o} \cdot \frac{1}{2}(1+z)^2,$$

when $q_o = 0$

Conclusion

Finally, we may remark that while we have been considering the above equations in the context of a cosmological model, one can apply them in other situations when a fluid approximation is appropriate.

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