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Fractional Derivative, A Novel Approach

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Abstract: Since various types of, fluid mechanics equations involves parameters which are not bounded at least the space defining variable viz. ocean current, smoke from cigarette etc. When solving these equations we apply various types of boundary conditions, subjected to these boundary conditions, smooth solutions are available if subjected boundary conditions permits solutions in the form of harmonic, algebraic and transcendental functions viz. trigonometric, polynomial, rational, exponential and logarithmic functions. However, a dynamical system rich of chaos is not bound to follow these functions. For e.g. take derivative $\frac{d\sin x}{dx} = \cos x$, notice the change in parameter x and its degrees, for Taylor's series of

$$\cos x = 1 - x^2/2! + x^4/4! - \ldots$$

operations of integer order derivative gives

$$\sin x = x - x^3/3! + x^5/5! - \ldots$$

Therefore an odd function changes to even function by operating integer order derivative operator, what are intermediary states? During its course of motion fluid need not to switch from one specific state to another but it must pass on through all intermediary states in contiguity. Thus any complete solution must reflect all possible states. This article is an attempt to define a smooth change representing all intermediary states through a precise fractional derivative for $0 < \alpha < 1$.

1. Introduction

In applied Mathematics and Mathematical analysis, a fractional derivative of arbitrary order, real or complex, first appeared in a letter to Guillaume de I'Hŏpital by Gottfried Wilhelm Leibnitz in 1695 [1]. Later on contributed by Neils Henrik Abel [2], Liouville [3], Oliver Heaviside [4], and foundation of the subject was laid by Liouville [5].

Some definitions of fractional derivatives and integrals

Let us assume that $f(x)$is a monomial of the form $f(x) = x^k$.The first derivative is as usual

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$$f'(x) = \frac{df(x)}{dx} = kx^{k-1}.$$

More general results gives

$$f^{(a)}(x) = \frac{d^a f(x)}{dx^a} = \frac{k!}{(k-a)!} x^{k-a}; a \in N \quad (1.1)$$

after replacing the factorials with the gamma function,

$$\frac{d^a x^k}{dx^a} = \frac{\Gamma(k+1)}{\Gamma(k-a+1)} x^{k-a}; k > 1$$

For $k = 1$ and $a = (1/2)$, of course a is replaced by any real or complex number for fractional order derivative.

$$\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}1}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{\frac{1}{2}} = \frac{2}{\sqrt{(\pi)}} x^{\frac{1}{2}}; k > 1.$$

For a general function $f(x)$ and $0 < \alpha < 1$, the complete fractional derivative is

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}} dt.$$

Riemann--Liouville fractional derivative

Computing nth order derivative over the integral of order $(n-\alpha)$,the α order derivative is obtained. It is important to remark that n is the smallest integer greater than α (that is, $n = [\alpha]$).

$${}_aD_t^{\alpha} f(t) = \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} \, {}_aD_t^{\alpha-n} f(t) = \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} \, {}_aI_t^{\alpha-n} f(t).$$

Caputo fractional derivative: Caputo's definition is illustrated as follows, where again $n = [\alpha]$

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(\mathrm{n})}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau.$$

Caputo's derivative does not require fractional order initial condition while solving differential equations.

Caputo--Fabrizio fractional derivative: Without singular kernel is given as

$${}_a^{CF}D_t^{\alpha} f(t) = \frac{1}{(1-\alpha)} \int_a^t f'(\tau) exp(-\alpha \frac{(t-\tau)}{(1-\alpha)}) \, d\tau.$$

Nature of fractional derivatives

The α^{th} derivative of a function $f(x)$ at a point x is alocal property only when α is an integer; this is not the true for non-integer power derivatives. Thus, a non-integer fractional derivative of a function $f(x)$ at $x = a$depends on all values of f, even those far away from α. Therefore, it is expected that the fractional derivative operation involves some sort of boundary conditions.

Fractional integral

Riemann--Liouville fractional integral:

$$ {}_aD_t^{-\alpha} f(t) = {}_aI_t^{-\alpha} f(t) = \frac{1}{(\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau. $$

Hadamard fractional integral:

$$ {}_aD_t^{-\alpha} f(t) = {}_aI_t^{-\alpha} f(t) = \frac{1}{(\Gamma(\alpha)} \frac{1}{(\Gamma(\alpha)} \int_a^t (\log \frac{t}{\tau})^{\alpha-1} f(\tau) \frac{d\tau}{\tau}. $$

Based on above formulae and approaches we define another form of fractional derivatives and integrals, as all of the above form of definition involves evaluation of gamma function at different real or complex number. Since evaluation of gamma function other than integers and 1/2, is a tedious task. Involvement of Gamma function also does not gives any insight about evolution of system. To short out this problem, change approach from generalizing outcomes of integer order derivatives to action oriented approach.

2. Main Results

Theorem 1.*Formula for fractional derivatives based on effect of action*

$$ D^{\frac{1}{k}} x^m = exp\left(\lim_{r\to 1} \int_{m-\frac{1}{k}}^{m} \frac{i}{e^{-2\pi i t}-1} \left(\int_0^{2\pi} \frac{e^{it\theta}}{1-re^{-i\theta}} d\theta\right) dt\right). x^{m-\frac{1}{k}} $$

Proof. Instead of looking at outcome of action of integer order derivatives, we focus on action of fraction order of derivative operator.Take $f(x) = x^{\mathrm{n}}$ i.e. monomial of degree n, and D be derivative operator, suppose $D^{\frac{1}{k}}$th, order derivative acts on x^{n}, and gives

$$ D^{\frac{1}{k}} x^n = \psi(n) x^{n-(1/k)}. \tag{2.1} $$

Again apply operator $D^{\frac{1}{k}}$on both side of equation (2.1), and get

$$ D^{\frac{1}{k}} D^{\frac{1}{k}} x^{\mathrm{n}} = D^{\frac{2}{k}} x^{\mathrm{n}} = \psi(n) D^{\frac{1}{k}} x^{n-(1/k)} = \psi(n)\psi(n-(1/k)) x^{n-(2/k)}. $$

where, ψ, is function of degree of x,at the time of operator takes action on it.Now suppose $D^{\frac{1}{k}}$ operated on both side of equation (2.1)k times then we get

$$ D^{\frac{1}{k}} \ldots . D^{\frac{1}{k}} x^{\mathrm{n}} = D x^{\mathrm{n}} = \psi(n)\psi(n-(1/k)) \ldots \psi(n-((k-1)/k)) x^{\mathrm{n}-1}. \tag{2.2} $$

Since $Dx^{\mathrm{n}} = nx^{\mathrm{n}-1}$, compare it with equation (2.2) and get

$$ \psi(n)\psi\left(n-\left(\frac{1}{k}\right)\right) \ldots \psi\left(n-\left(\frac{k-1}{k}\right)\right) = n $$

Take logarithm on both side to get

$$ \sum_{i=0}^{k-1} log\psi(n-(i/k)) = log\, n. \tag{2.3} $$

Multiply on both side of equation (2.3) by $(1/k)$ to get

$$\sum_{i=0}^{k-1}(1/k)\log\psi(n-(i/k)) = \left(\frac{1}{k}\right)\log n. \quad (2.4)$$

Take $k \to \infty$, and put $(i/k) = \mu$; $(1/k) = d\mu$ then summation in equation (2.4) changes to integral as below

$$\int_0^1 \log\psi(n-\mu)d\mu = (\log n)\int_0^1 d\mu = \log n.$$

Put $n - \mu = v$, then integral changes to

$$\int_{n-1}^{n} \log\psi(v)dv = \log n. \quad (2.5)$$

Differentiate equation (2.5) with respect to n, by using Leibnitz rule of differentiation under integral sign, to get

$$\log\psi(n) - \log\psi(n-1) = (1/n). \quad (2.6)$$

Solve equation (2.6) to get

$$\log\psi(n-(1/k)) = \left(\lim_{r\to 1}\int_{m-\frac{1}{k}}^{m}\frac{i}{e^{-2\pi it}-1}\left(\int_0^{2\pi}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).x^{m-\frac{1}{k}}$$

Thus we get

$$D^{\frac{1}{k}}x^m = \exp\left(\lim_{r\to 1}\int_{m-\frac{1}{k}}^{m}\frac{i}{e^{-2\pi it}-1}\left(\int_0^{2\pi}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).x^{m-\frac{1}{k}}$$

Now we can replace $(1/k)$ byα, $0 < \alpha < 1$, to get arbitrary order fractional derivative.

Corollary 1. *Corresponding fractional order integral is derived to be*

$$D^{-\frac{1}{k}}x^m = I^{\frac{1}{k}}x^m = \exp\left(-\lim_{r\to 1}\int_{m-\frac{1}{k}}^{m}\frac{i}{e^{-2\pi it}-1}\left(\int_0^{2\pi}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).x^{m-\frac{1}{k}}$$

Fractional derivative for special functions

$$D^{\frac{1}{k}}e^x = \sum_{m=0}^{\infty}\exp\left(\lim_{r\to 1}\int_{m-\frac{1}{k}}^{m}\frac{i}{e^{-2\pi it}-1}\left(\int_0^{2\pi}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right)\frac{x^{m-\frac{1}{k}}}{m!}$$

And corresponding fractional integral

$$.D^{-\frac{1}{k}}e^x = I^{\frac{1}{k}}e^x = \sum_{m=0}^{\infty}\exp\left(-\lim_{r\to 1}\int_{m-\frac{1}{k}}^{m}\frac{i}{e^{-2\pi it}-1}\left(\int_0^{2\pi}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right)\frac{x^{m+\frac{1}{k}}}{m!}e.$$

Remark 1.*Likewise we could define fractional derivative and integral to other special functions.*

Remark 2.*Thisapproach could be easily applied and be replaced in place of usual formulae we use in fluid mechanics.*

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