Hardy spaces on the disk and its applications

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Abstract
In this paper, we discuss the Hardy Hilbert Space on the open disk with center origin and radius unity. We have proved that $H^2$ Space is isomorphic to proper subspace of $L^2$ Space which has various applications in Quantum Mechanics.

Keywords: Lebesgue, Parseval Identity, Separable, orthonormal

1 Preliminaries

1.0.1 Definition (Inner Product Space)
An inner product space is a vector space $W$ (over field $K = \mathbb{R}$ or $\mathbb{C}$) with an inner product defined on it. Here, an inner product is a function $\langle , \rangle: W \times W \rightarrow K$ which satisfies the following properties:

1. $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u, u \rangle \geq 0$
4. $\langle u, u \rangle = 0 \Leftrightarrow u = 0$ (for all scalars $\alpha \in K$ and for all vectors $u,v,w \in W$)

Note 1: Every inner product space is a normed space with the norm induced by the inner product is given by

$$\sqrt{||u||^2} = \langle u, u \rangle$$

Note 2: An normed space $(W, ||\cdot||)$ is said to be complete if each Cauchy sequence converges in $W$.

1.1 Hilbert Space
An Hilbert Space is defined as the complete inner product space.
Example:

$$l^2 = \{(x_0, x_1, \ldots) : x_n \in \mathbb{C}, \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$$

i.e. all the elements of $l^2$ are the sequence of all the complex numbers that are square-summable. Inner product on $l^2$ is given by:

$$\langle (x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} x_n y_n$$

(it is an Hilbert sequence space)
1.2 Definition (Orthonormal sets and sequences)

An subset $X$ of an inner product space is said to be orthonormal if for all $u, v \in X$ we have,

$$< u, v > = \begin{cases} 0 & \text{if } u \neq v \\ \|u\|^2 & \text{if } u = v. \end{cases}$$

Note If norm of each element of an orthogonal set $X$ is 1 then the set is said to be orthogonal. i.e for all $u, v \in X$ we have,

$$< u, v > = \begin{cases} 0 & \text{if } u \neq v \\ 1 & \text{if } u = v. \end{cases}$$

1.3 Definition (Orthonormal basis)

An orthonormal subset $X$ of Hilbert space $W$ is said to be an orthonormal basis if span of $X$ is dense in $W$. i.e.

$$\text{Span } X = W$$

Note Every Hilbert space $W$ not equals to $\{0\}$ has an orthonormal basis.

1.4 Definition (Separable Hilbert Space)

A Hilbert-space $W$ is said to be separable if there exist a countable set which is dense in $W$.

Example: $l_2$ is a separable Hilbert space

Note Each orthonormal basis of an separable Hilbert space are countable. Therefore orthonormal basis of $l_2$ are countable

Recall

1. An orthonormal sequence $(e_n)$ is an orthonormal basis of a Hilbert - Space $W$

for all $u \in W$ we have

$$\sum_{n=0}^{\infty} |< u, e_n >|^2 = \|u\|^2$$

Parseval identity

2. Let $(e_n)$ be an orthonormal sequence in a Hilbert-space then

$$X = \sum_{n=0}^{\infty} X |\alpha_n|^2$$

converges in $W$ iff

the series

$$\sum_{n=0}^{\infty} \sum_{|\alpha_n|^2 < \infty}$$

converges in $\mathbb{R}$

2 THE HARDY-HILBERT SPACE

2.1 Definition

It is defined as the space of all the analytic functions which have a power series representation about origin with square-summable complex coefficients. It is denoted by $H^2$.

$$H^2 = \{ f : f(z) = \sum_{n=0}^{\infty} \alpha_n z^n : \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \}$$
Inner Product on $H^2$ is given by

$$< f, g >= \sum_{n=0}^{\infty} a_n \overline{b_n}$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $H^2$

**Theorem 2.1.** The Hardy-Hilbert space is a separable Hilbert Space.

**Proof:** Define an function;

$$\Phi : l^2 \rightarrow H^2$$
given by

$$\Phi(a_n)_{n=0}^{\infty} \rightarrow X a_n z^n$$

- **Φ is well defined** since $(a_n)_{n=0}^{\infty} \in l^2 \Rightarrow |a_n|^2 < \infty \Rightarrow \sum_{n=0}^{\infty} |a_n|^2 < \infty$ which being an power series is an analytic function whose coefficients are square summable hence is in $H^2$. ∴ $\Phi$ is well defined.

- **Clearly Φ is linear**

- **Φ is isometric**

  Fix $(a_n)_{n=0}^{\infty} \in l^2$ then we have

  $\Phi((a_n)_{n=0}^{\infty}) = \|\sum_{n=0}^{\infty} a_n z^n\|_{H^2} = \sum_{n=0}^{\infty} |a_n|^2 = \|(a_n)_{n=0}^{\infty}\|_{l^2}$

  ∴ $\Phi$ is an isometric.

- **since isometry property implies one one property**

  ∴ $\Phi$ is one one [1]

- **Φ is onto**

Let $f \in H^2$ then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$

define $x = (a_0, a_1, ...)$ Since

$$\|x\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

∴ $x \in l^2$

and

$$\Phi(x) = f$$

∴ $\Phi$ is onto.

Therefore $\Phi$ is an vector space isomorphism which also preserves the inner product.

Since $l^2$ is an separable Hilbert space hence $H^2$ is also an separable Hilbert Space.
Notations $D = \{ z : |z| < 1 \}$ denotes the open unit disk about origin in $C$ $S^1 = \{ z : |z| = 1 \}$ denotes the unit circle about origin in $C$.

**Theorem 2.2.** Radius of convergence of each function in $H^2$ is atleast 1

(i.e. each function in $H^2$ is analytic in the open unit disk $D$)

**Proof.** Let $z_0 \in D$ is fixed $\Rightarrow$ $|z_0| < 1$ $\Rightarrow$ the geometric series $\sum_{n=0}^{\infty} |z_0|^n$ converges. Let $f \in H^2$ is arbitrary. Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty$$

Since the series $\sum_{n=0}^{\infty} |a_n|^2$ converges $\Rightarrow |a_n|^2 \to 0 \Rightarrow |a_n| \to 0$

$\therefore \{ |a_n| \}_{n=0}^{\infty}$ is a convergent sequence hence bounded. $\therefore \exists M > 0$ such that

$$|a_n| \leq M \quad \forall \quad n \geq 0$$

Now

$$\sum_{n=0}^{\infty} |a_n z_n^n| \leq M \sum_{n=0}^{\infty} |z_0|^n$$

where being an geometric series right hand side converges.

$\therefore$ By Comparison test the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges absolutely. Since in Hilbert space absolute convergence implies convergence.

$\therefore$ the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges in $H^2$ since $z_0 \in H^2$ is arbitrary $\therefore$ each function in $H^2$ is analytic in the unit disk $D$.

2.2 **Definition ($L^2(S^1)$ space)**

It is defined as the space of all the equivalence classes of functions [4] that are Lebesgue measurable on $S^1$ and square integrable on $S^1$ with respect to Lebesgue measure normalized such that measure of $S^1$ is 1.

$$L^2(S^1) = \{ f : f \text{ is Lebesgue measurable on } S^1 \text{ and } \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta < \infty \}$$

Inner product on $L^2(S^1)$ is given by

$$< f, g > = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta}) d\theta$$

Note $L^2(S^1)$ is an Hilbert-space with the orthonormal basis given by $\{ e_n : n \in \mathbb{Z} \}$ where $e_n(e^{i\theta}) = e^{i\theta}$. Therefore

$$L^2(S^1) = \{ f : f = \sum_{n=\infty}^{n=-\infty} < f, e_n > e_n \}$$

2.2.1 **Definition ($Hc^2$ space)**

$$Hc^2 = \{ f \in L^2(S^1) : < f, e_n > = 0 \text{ for negative value of } n \}$$

$$Hc^2 = \{ f \in L^2(S^1) : f = \sum_{n=0}^{\infty} < f, e_n > e_n \}$$

$Hc^2$ is an subspace of $L^2(S^1)$ whose negative Fourier coefficients are 0.
\[
\{ e_n : n = 0, 1, \ldots \} \text{ are orthonormal basis of } H^2
\]

**Theorem 2.3.** \( H_c^2 \) is an Hilbert-space

**Proof.** Let \( f \in H_c^2 \) then there exist an sequence \( \{ f_n \}_{n=0}^{\infty} \) such that \( f_n \to f \) as \( n \to \infty \). Since \( f_n \in H_c^2 \) for all \( n \geq 0 \), we have:

\[
\langle f_n, e_k \rangle = 0 \quad \forall \ n \geq 0 \quad \text{and} \quad \forall \ k < 0
\]

Now for each \( k < 0 \) we have:

\[
|\langle f_n, e_k \rangle - \langle f, e_k \rangle| = |\langle f_n - f, e_k \rangle| \to 0 \quad \text{as} \quad n \to \infty \quad \text{(Schwarz Inequality [2])}
\]

Since \( k < 0 \) is arbitrary:

\[
\langle f, e_k \rangle = 0 \quad \forall \ k < 0
\]

Therefore \( H_c^2 \) is a closed subspace of \( L^2(S^1) \) Hence an Hilbert-Space

**Theorem 2.4.** The Hardy-Hilbert space can be identified as a subspace of \( L^2(S^1) \)

**Proof.** Define an function \( \psi : H^2 \to \mathbb{H}^2 \) where

\[
\hat{f}(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \hat{\psi}(z) = \sum_{n=0}^{\infty} a_n e_n
\]

- \( \psi \) is well defined

Let \( f \in H^2 \) Then \( \hat{f}(z) = \sum_{n=0}^{\infty} a_n z^n \) where \( \sum_{n=0}^{\infty} |a_n|^2 < \infty \).

Then by (recall 2) the series \( \hat{\psi} = \sum_{n=0}^{\infty} a_n e_n \) converges in \( H^2 \).

- \( \psi \) is well defined

- Clearly \( \psi \) is linear

For any arbitrary \( f \in H^2 \) where \( \hat{f}(z) = \sum_{n=0}^{\infty} a_n z^n \) we have:

\[
||\psi(f)|| = ||\hat{f}|| = \frac{1}{2\pi} \int_{0}^{2\pi} |\hat{f}(e^{i\theta})|^2 d\theta
\]

Now

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |\hat{f}(e^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{n=0}^{\infty} a_n e^{i\theta} \right)^2 d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{n=0}^{\infty} a_n e^{i\theta} \right) \left( \sum_{m=0}^{\infty} a_m e^{i\theta} \right)
\]
\[ \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^\infty \sum_{m=0}^\infty a_n \overline{a_m} e^{i(n-m)\theta} d\theta \]
\[ = \sum_{n=0}^\infty |a_n|^2 \quad \text{(since} \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} = \delta_{nm}) \]
\[ = ||f||^2 \]

Since \( f \in H^2 \) is arbitrary
\[ \therefore ||\psi(f)|| = ||f|| \quad \forall \ f \in H^2 \]

Therefore \( \psi \) is an isometry. Hence it preserves the inner product isometry \( \Rightarrow \) one one property.
\[ \therefore \psi \text{ is one one.} \]

- \( \psi \) is Onto

Let \( \tilde{f} \in \widehat{H^2} \). Then \( \tilde{f} = \sum_{n=0}^\infty <f, e_n> e_n \)

where \( <f, e_1>, <f, e_2>, ... \) are Fourier coefficients of \( f \) with respect to the orthonormal basis \( \{e_n : n \in N\} \).

Then by Parseval relation we have
\[ \sum_{n=0}^\infty |<f, e_n>|^2 = ||f||^2 < \infty \]

\[ a_n = <f, e_n> \quad \forall \ n \geq 0 \]

Since \[ \sum_{n=0}^\infty X_{n=0}^\infty Ca_n e^{i\theta n} \]
\[ \sum_{n=0}^\infty |<f, e_n>|^2 = ||f||^2 < \infty \]
\[ \sum_{n=0}^\infty |a_n|^2 = ||f||^2 < \infty \]

That is for each \( \tilde{f} \in \widehat{H^2} \) there exist \( f \in H^2 \) such that \( \psi(f) = \tilde{f} \)

Therefore \( \psi \) is onto

That is \( \psi \) is a vector space isomorphism which also preserves the norm. Therefore \( H^2 \) can be identified as a subspace of the \( L^2(S^1) \) space

\[ \square \]

### 3 Applications

1. In the mathematical rigourous formulation of Quantum Mechanics, developed by \textbf{Joh Von Neumann} the position and momentum states for a single non relavistic spin 0 Particle is the space of all the square integrable functions (\( L^2 \)). But \( L^2 \) have some undesirable properties and \( H^2 \) is much well behaved space so we work with \( H^2 \) instead of \( L^2 \).
References


