ON WEAKER FORM CONNECTEDNESS WITH RESPECT TO AN IDEAL

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Abstract: An ideal on a set X is a nonempty collection of subsets of X with heredity property which is also closed under finite unions. In this article we introduce the concept of ideal-connected spaces using ideals, called \( \mathcal{I} \)-connected spaces and extend some important results on connectedness to \( \mathcal{I} \)-connectedness. Also, we introduce the concept of Strongly ideal-connected spaces using ideals, called Strongly \( \mathcal{I} \)-connected spaces and extend some important results on Strongly connectedness to \( \mathcal{I} \)-connectedness. Conditions on \( \mathcal{I} \) are obtained under which connectedness, \( \mathcal{I} \)-connectedness and strongly \( \mathcal{I} \)-connectedness are equivalent.

Keywords: \( \mathcal{I} \)-connected, *-closure, ideal component, strongly \( \mathcal{I} \)-connectedness.

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1.INTRODUCTION

In1945, R. Vaidyanathaswamy [13] introduced the concept of ideal topological spaces. T. R. Hamlett, D. A. Rose [4] defined the local function and studied some topological properties using local function in ideal topological spaces in 1990. Since then many mathematicians studied various topological concepts in ideal-topological spaces. The first unified and extensive study on \( \tau \ast \)-topologies was done by Jankovic and Hamlett in [2] and proofs for the facts stated above may be found in [6]. The initial important articles on topological spaces are [4] and [5], a thesis [3] and a book that includes ideal is [12]. In this article we introduce the concept of ideal-connected spaces using ideals, called \( \mathcal{I} \)-connected spaces and extend some important results on connectedness to \( \mathcal{I} \)-connectedness.

2. PRELIMINARIES

Given a nonempty set X, a collection \( \mathcal{I} \) of subsets of X is called an ideal if,

(i) \( A \in \mathcal{I} \) and \( B \subseteq A \) implies \( B \in \mathcal{I} \) (heredity)

(ii) \( A \in \mathcal{I} \) and \( B \in \mathcal{I} \) implies \( A \cup B \in \mathcal{I} \) (additivity)

If \( X \not\in \mathcal{I} \), then \( \mathcal{I} \) is called a proper ideal. An ideal \( \mathcal{I} \) is called a \( \sigma \)-ideal if the following holds:

If \( \{A_n : n = 1,2, \ldots\} \) is a countable sub collection of \( \mathcal{I} \), then \( \bigcup \{A_n : n = 1,2, \ldots\} \in \mathcal{I} \)

The notation \((X,\tau, \mathcal{I})\) denotes a nonempty set X, a topology \( \tau \) on X and an ideal \( \mathcal{I} \) on X. Given a point \( x \in X \), \( N(x) \) denotes the neighbourhood system of \( x \); that is, \( N(x) = \{U \in \tau : x \in U \} \). \( \phi(X) \) denotes the collection of all subsets of X. Given space \((X,\tau, \mathcal{I})\) and a subset \( A \) of \( X \), we define

\[ A^*(\mathcal{I},\tau) = \{x \in X : U \cap A \not\in \mathcal{I}, \text{for every } U \in N(x)\} \]
We simply write $A^*$ for $A^*(\mathcal{I}, \tau)$, when there is only one ideal $\mathcal{I}$ and only one topology $\tau$ under consideration. If we define $cl^*(X)$ as, $cl^*(A) = A \cup A^*$, for all $A \in \mathcal{I}(X)$, then $cl^*$ is a Kuratowski closure operator. The topology determined by this closure operator is denoted by $\tau^*(\mathcal{I})$. $\tau^*(\mathcal{I}) = \{U \subseteq X : \tau(U) = \tau\}$ is a basis for $\tau^*(\mathcal{I})$. For every subset $A$ of a given topological space $(X, \tau, \mathcal{I})$, the sets $cl(A)$ (or $\overline{A}$) and $cl^*(A)$ will denote closure of $A$ with respect to $\tau$ and $\tau^*$ respectively.

3. CONNECTED SPACES

Let us start with a definition for $\mathcal{I}$-connected spaces.

**Definition: 3.1** Let $(X, \tau)$ be a topological space with an ideal $\mathcal{I}$ on $X$. A subset $Y$ of $X$ is said to be $\mathcal{I}$-connected if $Y \neq A \cup B$, $A,B \notin \mathcal{I}$ such that $\overline{A} \cap B = \phi = A \cap \overline{B}$.

**Remark :**3.1 Every connected set is $\mathcal{I}$-connected. We give the following example to show that the converse need not be true.

**Example :**3.1 Let $(R,\tau)$ denote the real line with the usual topology and $\mathcal{I}$ denote the ideal of all finite subsets of $X$. Let $Y = [0,2] \cup \{3,4,5\}$. Then $Y$ is $\mathcal{I}$-connected but not connected.

**Remark:**3.2 Let $(X, \tau)$ be a topological space with an ideal $\mathcal{I}$ on $X$. Let $X$ be $\mathcal{I}$-connected. If $\mathcal{J}$ is an ideal on $X$ with $\mathcal{I} \subseteq \mathcal{J}$, then $X$ is $\mathcal{J}$-connected.

We obtain equivalent conditions for a space to be a $\mathcal{I}$-connected space in the following theorem.

**Theorem: 3.2** Let $(X,\tau)$ be a topological space. Then the followings are equivalent.

(i) $X$ is $\mathcal{I}$-connected.

(ii) $X$ cannot be expressed as a union of two disjoint non-Ideal open sets.

(iii) $X$ cannot be expressed as a union of two disjoint non-Ideal closed sets.

**Proof:** (i) $\Rightarrow$ (ii)

Suppose (ii) is not true, then $X = A \cup B$, for some subset $A, B \notin \mathcal{I}$ such that $A, B$ are open and $A \cap B = \phi$. Then $A = \overline{A}$ and $B = \overline{B}$ so that $\overline{B} \cap A = \phi = \overline{A} \cap B$ This contradicts (i). Therefore (ii) is true.

(ii) $\Rightarrow$ (iii): Suppose (iii) is false. Then $X = A \cup B$, for some subsets $A,B \notin \mathcal{I}$ such that $A,B$ are closed and $A \cap B = \phi$. Then $X = A \cup B$, where $A, B \notin \mathcal{I}, A \cap B = \phi$ and $A = X - B, B = X - A$ are open. This contradicts (ii). Therefore (iii) is true.

(iii) $\Rightarrow$ (i): Suppose $X$ is not $\mathcal{I}$-connected. Then $X = A \cup B$, for some subsets $A,B \notin \mathcal{I}$, such that $\overline{A} \cap B = \phi = A \cap \overline{B}$. Then $\overline{A} \subseteq A$ and $\overline{B} \subseteq B$. Hence $X = A \cup B$, where $A, B \notin \mathcal{I}, A \cap B = \phi$ and $A, B$ are closed; which is a contradiction to (iii). So $X$ is $\mathcal{I}$-connected.

**Remark :**3.3 Let $(X, \tau)$ be a topological space and $\mathcal{I}$ be an ideal on $X$. A subset $Y$ of $X$ is $\mathcal{I}$-connected if and only if it is not possible to find open sets $A$ and $B$ in $X$ such that

(i) $Y \subseteq A \cup B$

(ii) $Y \cap A \notin \mathcal{I}$, $Y \cap B \notin \mathcal{I}$

(iii) $Y \cap \overline{A} \cap B = \phi$

(iv) $Y \cap A \cap \overline{B} = \phi$

We know that if $\{A_{\alpha} : \alpha \in \Lambda\}$ is a collection of connected subsets of a space $(X, \tau)$ such that $\cap A_{\alpha} \neq \phi$, then $\cup A_{\alpha}$ is also connected in $(X, \tau)$. Can this result be extended to a $\mathcal{I}$-connectedness? We first have the following theorem to get a partial affirmative answer. However, we shall find an example that gives a negative answer to this questions.

**Theorem: 3.3** Let $A_1$ and $A_2$ be two $\mathcal{I}$-connected sets with $A_1 \cap A_2 \notin \mathcal{I}$. Then $A_1 \cup A_2$ is $\mathcal{I}$-connected.
Proof: Suppose $A_1 \cup A_2$ is not $\mathcal{I}$-connected. Then $A_1 \cup A_2 = C \cup D$ where $C, D \notin \mathcal{I}$ and $(A_1 \cup A_2) \cap \overline{C} \cap D = \emptyset = C \cap \overline{D} \cap (A_1 \cup A_2)$. We have $A_1 \cap A_2 = (C \cap A_1 \cap A_2) \cup (D \cap A_1 \cap A_2) \notin \mathcal{I}$. Suppose $C \cap A_1 \cap A_2 \notin \mathcal{I}$, then $C \cap A_1 \notin \mathcal{I}$ and $C \cap A_2 \notin \mathcal{I}$. So either $C \cap A_1 \cap A_2 \notin \mathcal{I}$ or $D \cap A_1 \cap A_2 \notin \mathcal{I}$. Thus $A_1 \cap A_2$ is $\mathcal{I}$-connected, hence $A_1 \cup A_2$ is $\mathcal{I}$-connected.

Corollary: 3.4 The finite union of $\mathcal{I}$-connected sets $\{ A_1, A_2, \ldots, A_n \}$ for which $\bigcap_{i=1}^{n} A_i$ is a non-ideal set, is also an $\mathcal{I}$-connected set.

But arbitrary union of $\mathcal{I}$-connected sets $\{ A_i \}$, whose intersection $\bigcap_{i=1}^{n} A_i$ is a non-ideal set need not be $\mathcal{I}$-connected. The following example justifies this statement.

Example: 3.2 Let $X$ be the real line with the usual topology $\tau$. Let $A_0 = (0,1) \cup \{ n + 1 \}$, for all $n = 1, 2, \ldots$, and let $\mathcal{I}$ be the ideal of all finite subsets of $X$. Then $A_0$ is an $\mathcal{I}$-connected. Also $\bigcap_{i=1}^{\infty} A_i$ is a non-ideal set. However, $\bigcup_{i=1}^{\infty} A_i = (0,1) \cup \{ 2, 3, \ldots \}$ is not $\mathcal{I}$-connected.

Theorem: 3.5 Let $(X, \tau)$ be a topological space with an ideal $\mathcal{I}$ on $X$. If $A \subseteq X$ is $\mathcal{I}$-connected and $A \subseteq B \subseteq \overline{\text{cl}^* (A)}$ (closure of $A$ in $\tau^*$), then $B$ is $\mathcal{I}$-connected.

Proof: Suppose $B$ is not $\mathcal{I}$-connected. Then $B = C \cup D$, where $C, D \notin \mathcal{I}$ and $B \cap \overline{C} \cap D = \emptyset = C \cap \overline{D} \cap B$. Now $A = (A \cap C) \cup (A \cap D)$. Since $A$ is $\mathcal{I}$-connected, either $A \cap C \in \mathcal{I}$ or $A \cap D \in \mathcal{I}$. Suppose $A \cap D \in \mathcal{I}$ and let $x \in D - A$. Then for every neighbourhood $V$ of $x$, $V \cap A \notin \mathcal{I}$. In particular $V \cap A \cap C \notin \mathcal{I}$, which is contradiction to $D \cap \overline{C} \cap D = \emptyset$. Hence $B = A = \emptyset$, i.e., $D \subseteq A$. Therefore $D = \emptyset \subseteq A \in \mathcal{I}$, which is contradiction. Thus $B$ is $\mathcal{I}$-connected.

The above theorem is not true, if we replace $\mathcal{I}$-closed with closure. We give the following example to justify this fact.

Example: 3.3 Let $X$ be the real line with the usual topology. Let $A = [0,1] \cup \{ x : x \text{ is rational}, 4 < x < 5 \}$ and let $\mathcal{I}$ be the ideal of zero measurable sets. Then $A$ is $\mathcal{I}$-connected, but $\overline{A} = \text{cl}^* (A) = [0,1] \cup [4,5]$ is not $\mathcal{I}$-connected. We know that the continuous image of connected set is connected. We generalize this in the following theorem.

Theorem: 3.6 Let $f: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a continuous surjection. If $(X, \tau)$ is $\mathcal{I}$-connected, then $(Y, \sigma)$ is $f(\mathcal{I})$-connected, where $f(\mathcal{I}) = \{ f(1) : 1 \in \mathcal{I} \}$

Proof: Let $f: (X, \sigma, \mathcal{I}) \to (Y, \sigma)$ be a continuous surjection map and $X$ is $\mathcal{I}$-connected. Assume that $Y$ is not $f(\mathcal{I})$-connected, then $Y = B \cup C$, where $B, C \notin f(\mathcal{I})$, $B \cap C = \emptyset$ and $B, C$ are open.

Since $f$ is continuous, $f^{-1}(B)$, $f^{-1}(C)$ are open and $f^{-1}(B) \cap f^{-1}(C) = f^{-1}(B \cap C) = f^{-1}(\emptyset) = \emptyset$. Also $f^{-1}(B), f^{-1}(C) \notin \mathcal{I}$ (if $f^{-1}(B) \in \mathcal{I}$, then $B \in f(\mathcal{I})$, gives contradiction). Now $X = f^{-1}(B) \cup f^{-1}(C)$, where $f^{-1}(B), f^{-1}(C)$ are open, $f^{-1}(B) \cap f^{-1}(B) = \emptyset$ and $f^{-1}(B), f^{-1}(C) \notin \mathcal{I}$. Hence $X$ is not $\mathcal{I}$-connected; a contradiction to our assumption. Thus $Y$ is $f(\mathcal{I})$-connected.

In the following lemma, we show that extensions of $\mathcal{I}$-connected spaces by members of $\mathcal{I}$ are $\mathcal{I}$-connected.

Lemma: 3.7 Let $(X, \tau)$ be a topological space with an ideal $\mathcal{I}$ on $X$. Let $A, B \subseteq X$. If $A$ is $\mathcal{I}$-connected and $B \in \mathcal{I}$ then $A \cup B$ is $\mathcal{I}$-connected. (In particular, Let $(X, \tau)$ be a topological space with an ideal $\mathcal{I}$ on $X$. Let $A \subseteq X$. If $A$ is $\mathcal{I}$-connected and $X - A \in \mathcal{I}$, then $X$ is $\mathcal{I}$-connected.)

Proof: If $A \cup B$ is not $\mathcal{I}$-connected, then there exist open sets $C$ and $D$ in $X$ such that $A \cup B \subseteq C \cup D$ and

(i) $(A \cup B) \cap C \notin \mathcal{I}$, $(A \cup B) \cap D \notin \mathcal{I}$.
(ii) $(A \cup B) \cap (\overline{C} \cap D) = \emptyset$, $(A \cup B) \cap (C \cap \overline{D}) = \emptyset$
As \( B \in \mathcal{J} \), we have \( A \cap C \not\in \mathcal{J} \) (as \( B \cap C \in \mathcal{J} \)) and \( A \cap D \not\in \mathcal{J} \).

As \( A = (A \cap C) \cup (A \cap D) \), which is a contradiction to \( \mathcal{J} \)-connectedness of \( A \). Hence \( A \cup B \) is \( \mathcal{J} \)-connected.

We have already given example for \( \mathcal{J} \)-connected spaces which are not connected. But if the ideal \( \mathcal{J} \) satisfies some extra conditions, then we can expect that the space is connected if only if it is \( \mathcal{J} \)-connected. In the following theorem we show that if the ideal is a \( \mathcal{J} \)-boundary ideal i.e. \( \mathcal{J} \cap \tau = \{\phi\} \), then concept of connectedness and \( \mathcal{J} \)-connectedness coincide.

**Theorem: 3.8** Let \((X, \tau)\) be a topological space with an ideal \( \mathcal{J} \) on \( X \). If \( X \) is \( \mathcal{J} \)-connected and \( \mathcal{J} \cap \tau = \{\phi\} \), then \( X \) is connected.

**Proof:** Suppose \( X \) is not connected, then \( X = A \cup B \), where \( A \neq \phi, B \neq \phi \) and \( \overline{A \cap B} = \phi = A \cap \overline{B} \). Since \( \mathcal{J} \cap \tau = \{\phi\} \), we have \( A, B \not\in \mathcal{J} \). So, \( X \) is not \( \mathcal{J} \)-connected, a contradiction. Thus \( X \) is connected.

**Theorem: 3.9** Let \((X, \tau)\) be an \( \mathcal{J}_1 \)-connected space and let \((Y, \sigma)\) be an \( \mathcal{J}_2 \)-connected space. Assume that \( \mathcal{J}_1 \cap \tau \) is closed under arbitrary unions. If \( \mathcal{J} \) is an ideal such that \( P_i^{-1}(\mathcal{J}) \subset \mathcal{J} \), \( i = 1,2 \), then \( X \times Y \) is \( \mathcal{J} \)-connected.

**Proof:** If \( X \times \mathcal{J}_3 \), then \( X \times Y \) is in the ideal \( \mathcal{J} \) and hence \( X \times Y \) is \( \mathcal{J} \)-connected so assume that \( X \not\in \mathcal{J}_1 \).

Assume that \( X \times Y \) is not \( \mathcal{J} \)-connected, then \( X \times Y = A \cup B \), where \( A, B \not\in \mathcal{J} \), \( A \cap B = \phi \) and \( A, B \) are open in \( X \times Y \).

To each \( y \in Y \), define \( A_y = \{ x \in X : (x,y) \in A \} \) and \( B_y = \{ x \in X : (x,y) \in B \} \).

Let \( C = \{ y \in Y : A_y \in \mathcal{J}_3 \} \) and \( D = \{ y \in Y : B_y \in \mathcal{J}_3 \} \).

Then \( X = A_y \cup B_y \). To each \( y \), both \( A_y \) and \( B_y \) are open and \( A_y \cap B_y = \phi \). As \( X \) is \( \mathcal{J}_3 \)-connected, either \( A_y \in \mathcal{J}_1 \) or \( B_y \in \mathcal{J}_1 \).

In fact, to each \( y \in Y \), exactly one of \( A_y \) and \( B_y \) belongs to \( \mathcal{J}_3 \).

Therefore \( Y = C \cup D \) and \( C \cap D = \phi \). Now we claim that \( C \) is closed. Fix \( y \in C \). If \( A_y \not\in \mathcal{J}_3 \), then \( A_y \neq \phi \). Since \( A \) is open, to each \( x \in A_y \), there exist neighbourhoods \( U_x \) of \( x \) and \( V_y \) of \( y \) such that \( (x,y) \in U_x \times V_y \subset A \). As \( y \in C \), there is one \( y' \in V_y \cap C \), so \( U_x \times \{y'\} \subset A \) and hence \( U_x \subset A_y \) and as \( A_y \in \mathcal{J}_3 \), we have \( U_x \in \mathcal{J}_3 \). Therefore \( A_y \subset \bigcup \{ U_x : x \in A_y \} \in \mathcal{J}_3 \) (by assumption).

Hence \( A_y \in \mathcal{J}_3 \) and hence \( y \in C \). Therefore \( C \) is closed. Similarly \( D \) is closed. Since \( Y \) is \( \mathcal{J}_2 \)-connected, we have \( C \in \mathcal{J}_3 \) or \( D \in \mathcal{J}_3 \).

**Case (i):** If \( C \in \mathcal{J}_3 \), then \( X \times C \subset \mathcal{J}_3 \). Take \( E = \bigcup \{ B_y : y \in D \} \in \mathcal{J}_1 \cap \tau \) (assumption), So \( E \times Y \in \mathcal{J}_3 \) and \( (X \times C) \cup (E \times Y) \in \mathcal{J}_3 \). Fix \((x,y) \in B \). If \( y \in C \), then \((x,y) \in X \times C \). If \( y \notin C \), then \( y \in D \) and \( x \in B_y \subset E \). Therefore \((x,y) \in E \times Y \). Hence \( B \subset (X \times C) \cup (E \times Y) \). So \( B \in \mathcal{J}_3 \), This contradicts the fact \( B \notin \mathcal{J}_3 \).

**Case (ii):** If \( D \in \mathcal{J}_3 \), then \( X \times D \in \mathcal{J}_3 \); and as in case (i), we obtain a contradiction. Thus \( X \times Y \) is \( \mathcal{J} \)-connected.

**Corollary: 3.10** Let \((X, \tau_1)\) be topological spaces with ideals \( \mathcal{J}_i \) on \( X \) respectively for \( i = 1,2,...,n \). Let \( X = \prod_{i=1}^{n} X_i \) and \( \mathcal{J} \) be an ideal such that \( P_i^{-1}(\mathcal{J}_i) \subset \mathcal{J}_i \), \( i = 1,2,...,n \).

If \( \{ \mathcal{J}_i \cap \tau_1 : i = 1,2,...,n-1 \} \) is closed under arbitrary unions and \( X_i \) is \( \mathcal{J}_i \)-connected, then \( X \) is \( \mathcal{J} \)-connected.

**Corollary: 3.11** Let \((X, \tau)\) be a connected space and \((Y, \sigma)\) be \( \mathcal{J}_2 \)-connected. If \( \mathcal{J} \) be an ideal containing \( P_2^{-1}(\mathcal{J}_2) \), then \( X \times Y \) is \( \mathcal{J} \)-connected.

**Proof:** Consider \( \mathcal{J}_1 = \{\phi\} \), then \( \tau \cap \mathcal{J}_1 \) is closed under arbitrary unions, so by theorem 2.9, \( X \times Y \) is \( \mathcal{J} \)-connected.

**Definition: 3.2** Let \((X, \tau)\) be a topological space and \( \mathcal{J} \) be an ideal in \( X \). A connected component \( C \) of \( X \) with respect to \( \tau \) is called \( \tau \)-connected component of \( X \). A \( \tau \)-connected component \( C \) of \( X \) is said to be an ideal component in \( X \) if \( C \in \mathcal{J} \).

**Example: 3.4** Let \( X = \bigcup \{ [0,1] \cup [2,3] \cup [4,5] \cup ... \cup [2n, 2n+1] \cup ... \} \) with the subspace topology induced by the usual topology of \( \mathbb{R} \). If \( \mathcal{J} \) is the set of all bounded subsets, then every component of \( X \) is an ideal component.

**Theorem: 3.10** Let \((X, \tau_1)\) be \( \mathcal{J}_1 \)-connected and \((Y, \tau_2)\) be \( \mathcal{J}_2 \)-connected. Assume that any union of ideal components is a member of \( \mathcal{J}_1 \). If \( \mathcal{J} \) is an ideal in \( X \times Y \) containing \( P_1^{-1}(\mathcal{J}_1) \) and \( P_2^{-1}(\mathcal{J}_2) \), then \( X \times Y \) is \( \mathcal{J} \)-connected.

**Proof:** If \( X \in \mathcal{J}_1 \), then \( X \times Y \) is in the ideal \( \mathcal{J} \) and hence \( X \times Y \) is \( \mathcal{J} \)-connected. So we assume that \( X \notin \mathcal{J}_1 \). Assume that \( X \times Y \) is not \( \mathcal{J} \)-connected. Then \( X \times Y = A \cup B \), where \( A, B \not\in \mathcal{J}_3 \), \( A \cap B = \phi \) and \( A, B \) are open sets. For every component \( C \) of \( X \) and \( D \) of \( Y \), \( C \times D \) is a connected subset of \( X \times Y \) and hence \( C \times D \subset A \) or \( C \times D \subset B \).
For every component $D$ of $Y$, write

$$A_D = \bigcup \{ C : C \text{ is a component of } X \text{ and } C \times D \subseteq A \}.$$  

$$B_D = \bigcup \{ C : C \text{ is a component of } X \text{ and } C \times D \subseteq B \}.$$  

Now we claim that $A_D$ is open. Let $x \in A_D$, then there exists a component $C$ of $X$ such that $x \in C$ and $C \times D \subseteq A$. Fix $y \in D$. Therefore $(x, y) \in C \times D \subseteq A$. Since $A$ is open, there exist neighbourhoods $U_x, V_y$ of $x, y$ respectively such that $U_x \times V_y \subseteq A$. If $x \in \text{cl}(B_D)$, then $U_x \cap B_D \neq \emptyset$. Let $x' \in U_x \cap B_D$ i.e. $x' \in U_x \cap C_0$, for some component $C_0$ where $C_0 \times D \subseteq B$. Let $(x', y) \in U_{x'} \times V_{y'} \subseteq B$, where $U_{x'}, V_{y'}$ are some neighbourhoods of $x', y$ respectively. Then $(x', y) \in (U_x \cap U_{x'}) \times (V_y \cap V_{y'}) \subseteq A \cap B$, which contradicts $A \cap B = \emptyset$. Therefore $x \in A_D$ implies that $x$ is not a limit point of $B_D$. That is, $A_D$ is open. Similarly $B_D$ is open. Thus $X = A_D \cup B_D$, and $A_D$, $B_D$ are open. So exactly one of $A_D$, $B_D$ is in $\mathfrak{A}_1$, because $X \notin \mathfrak{A}_1$.

Let $\mathfrak{A}_1 = \{ D \subseteq Y : D \text{ is component of } Y \text{ and } A_D \in \mathfrak{A}_1 \}$ and  

$$\mathfrak{A}_2 = \{ D \subseteq Y : D \text{ is component of } Y \text{ and } B_D \in \mathfrak{A}_1 \}.$$  

Write $D_1 = \bigcup_{D \in \mathfrak{A}_1} D$ and $D_2 = \bigcup_{D \in \mathfrak{A}_2} D$. Then $Y = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$.

We claim that $D_1$ is closed. Fix $d \in D_1$. Let $D$ be the component of $Y$ such that $d \in D$. Suppose $d \notin D$. Then $D \in \mathfrak{A}_1 \Rightarrow A_D \notin \mathfrak{A}_1 \Rightarrow B_D \notin \mathfrak{A}_1 \Rightarrow D \notin \mathfrak{A}_1$. By (1) and our assumption, there is a component $C$ of $X$ such that $C \notin \mathfrak{A}_1$, and $C \times D \subseteq A$. Fix a member $c \in C$. Then $(c, d) \in A$. Since $A$ is open, there exist neighbourhoods $U_c, V_d$ of $c, d$ respectively, such that $(c, d) \in U_c \times V_d \subseteq A$. So there is a member $d' \in V_d \cap D$ and there is a component $D'$ of $Y$ such that $d' \in D'$ and $A_D \in \mathfrak{A}_1$, so that $(c, d') \in (U_c \times V_{d'}) \subseteq A$. Therefore $(c, d') \subseteq A$ and $(c, d') \subseteq B$. Therefore $D_2$ is closed. Thus $Y = D_1 \cup D_2$, where $D_1$, $D_2$ are closed and $D_1 \cap D_2 = \emptyset$. Since $Y$ is $\mathfrak{A}_1$-connected, either $D_1 \in \mathfrak{A}_1$ or $D_2 \in \mathfrak{A}_1$. Without loss of generality, we assume that $D_1 \in \mathfrak{A}_1$.

Let $X \times D_1 \in \mathfrak{A}_1$. Take $E = \bigcup_{D \in \mathfrak{A}_1} D \subseteq \mathfrak{A}_1$ (by assumption). So $E \times Y \in \mathfrak{A}_1$ and hence $(X \times D_1) \cup (E \times Y) \in \mathfrak{A}_1$. It is enough to prove that $B \subseteq (X \times D_1) \cup (E \times Y)$. Fix $(x, y) \in B$. Then there exist components $C$ and $D$ such that $(x, y) \in C \times D \subseteq B$. If $y \in D_1$, then $(x, y) \in X \times D_1$. If $y \notin D_1$, then $y \in D_2$. Take $y' \in D_2 = \bigcup_{D \in \mathfrak{A}_2} D$ and hence $x \in C \subseteq B \subseteq E$, for some $D \in \mathfrak{A}_1$. Therefore $(x, y) \in E \times Y$. Hence $B \subseteq (X \times D_1) \cup (E \times Y) \in \mathfrak{A}_1$. This is a contradiction to $B \notin \mathfrak{A}_1$. Hence $X \times Y$ is $\mathfrak{A}_1$-connected.

4. STRONGLY $\mathfrak{A}_1$-CONNECTED SETS  

Let us begin with following definition.

**Definition:** Let $(X, \tau)$ be a topological space and let $\mathfrak{A}$ be an ideal on $X$. A subset $A$ of $X$ is said to be strongly $\mathfrak{A}$-connected if there is a $\tau$-connected subset $B$ of $X$ such that $A = B \cup C$, where $C \in \mathfrak{A}$.

Every connected set is strongly $\mathfrak{A}$-connected set, but converse need not be true. It follows from the following example.

**Example:** Let $(R, \tau)$ denote the set of real numbers with the usual topology and $\mathfrak{A}$ be the ideal of all finite subsets of $R$. Let $A = \{0, 2\} \cup \{3, 4, 5\}$. Then $A$ is strongly $\mathfrak{A}$-connected, but not connected.

The following theorem gives the relation between $\mathfrak{A}$-connectedness and strongly $\mathfrak{A}$-connectedness.

**Theorem:** Let $(X, \tau)$ be a topological space with a ideal $\mathfrak{A}$ on $X$. If $(X, \tau)$ is strongly $\mathfrak{A}$-connected, then it is $\mathfrak{A}$-connected.

**Proof:** Assume that $(X, \mathfrak{A})$ is strongly $\mathfrak{A}$-connected and $X = B \cup C$ where $B$ is $\tau$-connected and $C \in \mathfrak{A}$. Suppose $X = D_1 \cup D_2$, where $D_1$, $D_2$ are open and $D_1 \cap D_2 = \emptyset$. Then $B = (D_1 \cap B) \cup (D_2 \cap B)$ and $D_1 \cap B = \emptyset$ or $D_2 \cap B = \emptyset \Rightarrow D_1 \subseteq X \setminus B$ or $D_2 \subseteq X \setminus B \Rightarrow D_1 \subseteq C$ or $D_2 \subseteq C \Rightarrow D_1 \in \mathfrak{A}$ or $D_2 \in \mathfrak{A}$. Hence $X$ is $\mathfrak{A}$-connected.

The converse of the above theorem is not true.

**Example:** Let $X = \{0, 1, 1/2, 1/3, \ldots\}$ and $\tau$ be the topology denoted by the usual topology in $R$. Let $\mathfrak{A}$ be the ideal of all finite subsets. Then $X$ is $\mathfrak{A}$-connected. For if $X$ is not $\mathfrak{A}$-connected, then $X = B \cup C$, where $B, C \not\in \mathfrak{A}$. Then $B \cap C = \emptyset$ and $B, C$ are open.

Therefore $0 \in B$ or $0 \in C$, which implies that $C$ is finite or $B$ is finite, so that $C \in \mathfrak{A}$ or $B \in \mathfrak{A}$ which is a contradiction. But $X$ is not strongly $\mathfrak{A}$-connected because only connected subsets of $X$ are singletons whose compliments are not in $\mathfrak{A}$.  


Theorem: 4.2  Given (X, τ, 3 ) such that the  ideal 3 is a τ - boundary ideal. Then the following are equivalent:

(i) X is connected
(ii) X is 3-connected
(iii) X is strongly 3-connected

Remark: 4.1  Let (X, τ) be a topological space with a ideal 3 on X. Let A, B ⊆ X. If A is strongly 3-connected and B ∈ 3, then A ∪ B is strongly 3-connected.

Theorem: 4.3 Let Aᵢ (i = 1,2,…..n) be strongly 3-connected sets such that \( \bigcap_{i=1}^{n} A_i \notin 3 \), then \( \bigcup_{i=1}^{n} A_i \) is strongly 3-connected.

Proof: Since each Aᵢ (i = 1,2,…n) is strongly 3-connected, we have Aᵢ = Bᵢ ⊇ Cᵢ , where Bᵢ is connected and Cᵢ ∈ 3. As \( \bigcap_{i=1}^{n} A_i \notin 3 \), we get Aᵢ ⊈ 3, and Bᵢ ⊈ 3, for all j. Let Fⱼ = (\( \bigcap_{i=1}^{n} A_i \)) \cap Cⱼ \( \). Then Fⱼ ∈ 3, for all j.

Therefore \( \bigcup_{i=1}^{n} Fⱼ \) ∈ 3. Put E = ( \( \bigcap_{i=1}^{n} A_i \)) − ( \( \bigcup_{i=1}^{n} Fⱼ \)). Then E ⊈ 3, because \( \bigcap_{i=1}^{n} A_i \) ⊈ 3 and Fⱼ ∈ 3. Now E ⊆ Bⱼ for all j and hence E ⊆ \( \bigcap_{i=1}^{n} Bᵢ \notin 3 \). In particular \( \bigcap_{i=1}^{n} Bᵢ \notin 3 \).

Hence \( \bigcup_{i=1}^{n} Bᵢ \) is connected and hence \( \bigcup_{i=1}^{n} Aᵢ = ( \bigcup_{i=1}^{n} Bᵢ ) \cup \bigcup_{i=1}^{n} Cᵢ \) ∈ 3. Thus \( \bigcup_{i=1}^{n} Aᵢ \) is strongly 3-connected.

This theorem need not be true, if the family \( \{ Aᵢ \} \) is an infinite family whose intersection is a non ideal set, as this may be seen from the following example.

Example: 4.3 Let X be the real line with the usual topology τ. Let Aᵣ = (0,1) ∪ \( \{ n + 1 \} \) for all n = 1,2… and let 3 be the ideal of all finite subsets of X. Then Aᵣ is strongly 3-connected.

Also \( \bigcap_{n=1}^{∞} Aᵣ \) is a non ideal set. However, \( \bigcup_{n=1}^{∞} Aᵣ = (0,1) \cup \{ 2,3,…. \} \) is not strongly 3-connected.

It is well known that the continuous image of a connected set is connected. This result can be generalized as follows.

Theorem: 4.4 Let f: (X, τ, 3) → (Y, σ) be a continuous surjection. If (X, τ) is strongly 3-connected, then (Y, σ) is strongly f(3) – connected, where f(3) = \{ f(1): I ∈ 3 \}.

Proof: Let f: (X, τ, 3) → (Y, σ) be a continuous surjection and let (X, τ) is strongly 3-connected. Then X = B ∪ C, where B is connected and C ∈ 3. Therefore Y = f(X) = f(B ∪ C) = f(B) ∪ f(C), where f(B) is connected and f(C) ∈ f(3). Thus (Y, σ) is strongly f(3) connected.

Theorem: 4.5 If \( \widetilde{I} \in 3 \), for all \( I \in 3 \) then whenever A is strongly 3-connected, then B is also strongly 3-connected, for all B with A ⊆ B ⊆ \( \widetilde{A} \). In particular \( \widetilde{A} \) is strongly 3-connected, if \( \widetilde{I} \in 3 \), for all \( I \in 3 \).

Proof: Suppose A is strongly 3-connected. Then A = C ⊇ D, where C is connected and D ∈ 3. Since A ⊆ B ⊆ \( \widetilde{A} \) and A = C ⊇ D ⊆ B, we have B = (C ∩ B) ∪ (D ∩ B), where C ∩ B is connected as C = C ⊆ C and D ⊆ D ⊆ 3. Hence B is strongly 3-connected. As a particular case when A is strongly 3-connected, \( \widetilde{A} \) is strongly 3-connected, for all \( \widetilde{I} \in 3 \).

The condition of I ∈ 3 implies \( \widetilde{I} \in 3 \) can not be relaxed from the previous theorem 4.5. This is justified by the next example.
Example: 4.4  Let (R, τ) denote the real line with the usual topology and let  be the ideal of all with zero measure. Let \( A = [0,1] \cup \{ x : x \) is rational, \( 4 < x < 5 \}. \) Then A is strongly \( 3 \) - connected, but \( \bar{A} = [0,1] \cup [4,5] \) is not strongly \( 3 \) - connected.

Example: 4.5  Let \( X = [0,1] \cup \{2,3,4,5\} \) with the usual topology and let \( 3 \) be the ideal of all finite subsets of X. Then X is strongly \( 3 \)-connected, but \( X \times X \) is not strongly \( 3 \times 3 \) - connected.

Now we discuss strongly ideal connectedness of product of two strongly \( 3 \)-connected sets with a suitable ideal in the product space.

Theorem: 4.6  Let \( (X, \tau_1) \) be strongly \( 3_1 \)-connected and \( (Y, \tau_2) \) be strongly \( 3_2 \)-connected. If \( 3 \) is an ideal on \( X \times Y \) such that \( p_i^{-1}(3_i) \subset 3, i = 1,2 \), then \( X \times Y \) is strongly \( 3 \)-connected, where \( p_1 : X \times Y \to X, p_2 : X \times Y \to Y \) are the projections and \( p_i^{-1}(3_i) = \{ p_i^{-1}(I_i) : I_i \in 3_i, i=1,2 \} \)

Proof: Suppose X is strongly \( 3_1 \)-connected and Y is strongly \( 3_2 \)-connected. Then \( X = A \cup C_1 \) and \( Y = B \cup C_2 \), where A,B are connected subsets of X and Y respectively and \( C_1, C_2 \in 3 \).

Then \( X \times Y = (A \times B) \cup ((C_1 \times Y) \cup (X \times C_2)) \). Since A x B is connected with respect to the product topology \( \tau_1 \times \tau_2 \) and \( C_1, C_2 \in 3 \), we have \( (C_1 \times Y) \cup (X \times C_2) \in 3 \). Thus X x Y is strongly \( 3 \)-connected.

Corollary: 4.7  Let \( (X_i, \tau_i), i = 1,2,..,n \) be a topological space with a ideal \( 3_i \) on \( X_i \), for \( i = 1,2,3,....,n \). If each \( X_i, i = 1,2,....n \) is strongly \( 3_i \)-connected and if \( 3 \) is a ideal containing \( p_i^{-1}(3_i) \), then \( \pi_i X_i \) is strongly \( 3 \)-connected, where \( p_i : \pi_i X_i \to X_i \) are the projection and \( p_i^{-1}(3_i) = \{ p_i^{-1}(I_i) : I_i \in 3_i, i=1,2,3,....,n \} \)

Corollary: 4.8  Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be two topological spaces with ideals \( 3_1, 3_2 \) on X, Y respectively. Let \( (X, \tau_1) \) be a strongly \( 3_1 \)-connected and \( (Y, \tau_2) \) be \( 3_2 \)-connected. If \( 3 \) is a ideal containing \( p_1^{-1}(3_1) \) and \( p_2^{-1}(3_2) \), then \( X \times Y \) is \( 3 \)-connected.

Consider the following definition.

Definition: 4.2  Let \( (X, \tau) \) be a topological space with a ideal \( 3 \) on \( X \). A subset \( A \subseteq X \) is said to be \( 3 \)-well linked if \( \bar{A} \) is strongly \( 3 \)-connected.

From the theorem 2.9, it follows that if \( \bar{I} \in 3, \) for all \( I \in 3 \), then every strongly \( 3 \)-connected subsets of X are \( 3 \)-well linked. The converse of the observation is not true if \( \bar{I} \notin 3 \) for some \( I \in 3 \). This may be seen from the following example.

Example 4.6  Let \( Q \) be a set of all rational numbers and let \( 3 = \{ \phi \} \). Take \( X = Q \cup \{ \phi \} \). Then \( \bar{X} \) is strongly \( 3 \)-connected and hence X is \( 3 \)-well linked but X is not strongly \( 3 \)-connected.

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