Closed Classes of Monotone Boolean Functions

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Abstract

Boolean functions are one of the main objects of discrete mathematics. The class of monotone Boolean functions is pre-complete in the set of all Boolean functions, that is, there is no class of Boolean functions which is closed (with respect to superposition), contains all monotone functions, and differs both from the class of monotone Boolean functions and from the set of all Boolean functions. In the present article our aim is to discuss the basic structure and functions of closed classes of monotone Boolean functions.

Keywords : Monotone Boolean functions, *n*-tuples, inclusion-exclusion principle, Bonfierroni's inequalities.

Introduction

Monotone Boolean functions are an important object in discrete mathematics.

A Boolean function is a function whose variables take the values 0 and 1, as does the function itself.

It turned out later that the language of Boolean functions is convenient to describe the functioning of certain discrete control systems like contact networks, Boolean networks, logical networks, and so on. In the set of all Boolean functions, the following classes of functions are well known: linear functions, monotone functions, threshold functions, symmetric functions, and some other classes. Among these classes, the class of monotone Boolean functions is of significant interest. The class of monotone Boolean functions has the greatest cardinality among the classes listed above. The class of monotone Boolean functions is pre-complete in the set of all Boolean functions, that is, there is no class of Boolean functions which is closed (with respect to superposition), contains all monotone functions, and differs both from the class of monotone Boolean functions and from the set of all Boolean functions.

So, in this article our aim is to discuss the basic structure and functions of closed classes of monotone Boolean functions.

Closed classes of monotone Boolean functions

Let A be a set of Boolean functions. By the closure of the set A we mean the set of all Boolean functions representable by formulae in terms of functions in the set A. The closure of the set A is denoted by $[A]$.

A class A of Boolean functions is said to be closed if $A = [A]$.

Post [1] described all closed classes of Boolean functions. The description of these classes can also be found in the monographs [2], [3].

Among the closed classes of Boolean functions one can single out countably many classes consisting of monotone functions. In particular, one of these classes is the class of all monotone Boolean functions.

We now list all closed classes of monotone Boolean functions.

We say that a Boolean function $f(x_1,\ldots, x_n)$ preserves the constant 0 (the constant 1) if $f(0,\ldots,0) = 0$ ($f(1, \ldots,1) = 1$). A Boolean function with $f(x_1, \ldots, x_n) = \tilde{f}(\tilde{x}_1,\ldots,\tilde{x}_n)$ is said to be self-dual, that is, the function f takes different values at any two opposite n-tuples.

The following classes of monotone Boolean functions are closed:

- the set A_1 of all monotone Boolean functions;
- the set A_2 of all monotone Boolean functions preserving the constant 1;
- the set A_3 of all monotone Boolean functions preserving the constant 0;
- the sets A_4 of all monotone Boolean functions preserving the constants 0 and 1;
- the set F_2^m of all monotone Boolean functions f preserving the constants 0 and 1 and such that any m n-tuples on which f vanishes have at least one common zero component, $m = 2, 3, \ldots$;
- the set F_3^m of monotone Boolean functions f such that any m n-tuples on which f vanishes have at least one common zero component, $m = 2, 3, \ldots$;

- the set F_6^m of all monotone Boolean functions f preserving the constants 0 and 1 and such that any m n-tuples on which f is equal to 1 have at least one common component equal to 1, m = 2, 3,…;
- the set F_7^m of all monotone Boolean functions f such that any m n-tuples on which f is equal to 1 have at least one common component equal to 1, m = 2, 3.... ;
- the set F_2^∞ of all monotone Boolean functions f preserving the constants 0 and 1 and such that all n-tuples on which f vanishes have at least one common zero component;
- the set F_3^∞ of all monotone Boolean functions f such that all n-tuples on which f vanishes have at least one common zero component;
- the set F_6^∞ of all monotone Boolean functions f preserving the constants 0 and 1 and such that all n-tuples on which the function f is equal to 1 have at least one common component equal to 1;
- the set F_7^∞ of all monotone Boolean functions f such that all n-tuples on which the function f is equal to 1 have at least one common component equal to 1;
- the set D_2 of all self-dual monotone Boolean functions.

For an arbitrary set of Boolean functions H we denote by $H(n)$ the set of function in H depending on n variables, that is on $r_1,\ldots r_n$.

Since $A_1(n) = M(n)$, it follows that the quantity $|A_i(n)|$, $1\leq i \leq 4$, satisfies the asymptotic formulae. One can also readily see that as n64

$$|F_2^m(n)| \sim F_3^m(n)|, \quad |F_6^m(n)| \sim F_7^m(n)|,$$

for any $m \geq 2$, and also

$$|F_2^\infty(n)| \sim F_3^\infty(n)|, \quad |F_6^\infty(n)| \sim F_7^\infty(n)|.$$

Let us prove that

$$|F_3^\infty(n)| = |F_7^\infty(n)| \qquad (1.1)$$

for any $n \geq 1$ and that

$$|F_3^m(n)| = |F_7^m(n)| \qquad (1.2)$$

for any $m \geq 2$.

To this end, to any Boolean function $f\ (x_1,\ldots, x_n)$ we assign a Boolean function g_f $(x_1,\ldots, x_n)$ such that

$$g_f(\tilde{\alpha}) = \begin{cases} 0 & \text{if } f(\tilde{\tilde{\alpha}}) = 1, \\ 1 & \text{if } f(\tilde{\tilde{\alpha}}) = 1 \end{cases}$$

It is clear that this correspondence is one-to-one. At the same time, if f is a monotone function, then so is g_f. This holds because, if an n-tuple $\tilde{\alpha} = (\alpha_1,\ldots,\alpha_n)$ is a lower unit of a monotone Boolean function $f(x_1,\ldots,x_n)$, then the n-tuple $\tilde{\tilde{\alpha}} = (\bar{\alpha}_1,\ldots,\bar{\alpha}_n)$ is an upper zero of the function $g_f\,(x_1,\ldots, x_n)$.

$\Rightarrow$ the function $f \in F_3^\infty(n)$ if and only if the function $g_f \in F_7^\infty(n)$. Hence,

$$|F_3^\infty(n)| = |F_7^\infty(n)|$$

for any $n \geq 1$ and $|F_3^m(n)| = |F_7^m(n)|$ for any $n \geq 1$, $m = 2, 3, \ldots$

so we can treat only the sets $D_2(n)$, $F_7^\infty(n)$ for $m = 2, 3, \ldots$

Theorem 1. For odd n64

$$|F_7^\infty(n)| \sim n.2^{\binom{n}{(n-1)/2}} \exp\left\{ \binom{n-1}{(n-3)/2}(2^{-(n-1)/2-3} + n^2 2^{-n-4} - n2^{-n-2}) \right\},$$

and for even n64

$$|F_7^\infty(n)| \sim n.2^{\binom{n-1}{n/2}+1} \exp\left\{ \binom{n-1}{n/2+1}\left(2^{-n/2-1} - n^2 2^{-n-5} - 3n2^{-n-4}\right) + \binom{n-1}{n/2-1}(2^{-n/2} + n^2 2^{-n-3} - n2^{-n-2} \right\}.$$

The validity of Theorem follows easily from inclusion-exclusion principle with respect to the number of common components equal to 1 in then n-tuples in E^n at which the functions in $F_7^\infty(n)$ are equal to 1 and Bonfierroni's inequalities [4].

Theorem 2. For any fixed $m \geq 3$

$$\left|F_7^m(n)\right| \sim F_7^\infty(n)\right|,$$

as n64, that is, for odd n64 and $m \geq 3$

$$\left|F_7^m(n)\right| \sim n2^{\binom{n-1}{(n-1)/2}} \exp\left\{\begin{array}{l} \binom{n-1}{(n-3)/2}(2^{-(n-1)/2} \\ +n^2 2^{-n-4} - 2n2^{-n-2}) \end{array}\right\}$$

and for even n64 and $m \geq 3$

$$\left|F_7^2(n)\right| \sim n.2^{\binom{n-1}{n/2-1}} \exp\left\{\begin{array}{c} \binom{n-1}{n/2+1}(2^{-n/2-1} - n^2 2^{-n-5} \\ -3n2^{-n-4} + \binom{n-1}{n/2-1}(2^{-n/2} \\ +n^2 2^{-n-3}) - n2^{-n-2}) \end{array}\right\}$$

The above result was announced in [5] and [6].

Theorem 3. For odd n64

$$\left|D_2(n)\right| \sim 2^{\binom{n-1}{(n+1)/2}} \exp\left\{\begin{array}{c} \binom{n-1}{n-1/2}(2^{-(n-1)/2} - 3n^2 2^{-n-4} \\ -n2^{-n-3}) + \binom{n-1}{(n+3)/2} \\ (2^{-(n+3)/2} + n^2 2^{-n-6} - n2^{-n-5}) \end{array}\right\}$$

and for even n64

$$\left|D_2(n)\right| \sim 2^{\binom{n-1}{n/2}} \exp\left\{\begin{array}{c} \binom{n-1}{n-1/2}(2^{-n/2-1} - n2^{-n-4}) \\ +\binom{n-1}{n/2+1}(2^{-n/2-1} + n^2 2^{-n-5} - n2^{-n-4}) \end{array}\right\}$$

The assertion of above theorem follows easily from the equality [7] :

$$\left|D_2(n)\right| = \left|F_7^2(n-1)\right|$$

Which was diseocovered by Sapozhenko [8].

References

[1] E.L. Post, "The two-valued iterative systems of mathematical logic", (Ann. of Math. Stud., vol. 5) Princeton Univ. Press, Princeton 1941.

[2] S.S. Marchenkov, "Closed classes of Boolean functions". Fiz. Mat. Lit., Moscow

2000. (Russian)

[3] I. Wegener, "The complexity of Boolean functions", Wiley/Teubner, Chichester/Stuttgart, 1987.

[4] S. Bublitz, U. Schiirfeld, and I. Wegener, "Properties of complexity measures for PRAMs and WRAMs", Theoret. Comput. Sci. 48 (1986), 53-73.

[5] A.D. Korshunov, "On the cardinality and structure of some closed Post classes (of families of subsets of a finite set)", Dokl. Akad. Nauk SSSR 295 (1987), 533-537; English transl., Soviet Math. Dokl. 36 (1988), 88-91.

[6] A.D. Korshunov, "On the cardinality and structure of closed Post classes (of families of subsets of a finite set)", Modeli Metody Optim., (Trudy Inst. Mat. (Novosibirsk), vol. 10) Nauka, Novosibirsk 1988, p. 159-204. (Russian)

[7] A.D. Korshunov and I. Shmulevich, "The number of special monotone Boolean functions and statistical properties of stack filters", Diskret. Anal Issled. Oper. Ser. 1 7:3 (2000), 17-44. (Russian)

[8] A.A. Sapozhenko, "Method of boundary functionals in enumerative isoperimetric problems", Collection of Works of the Seminar on Discrete Mathematics and Its Applications (Moscow, Febr. 2-4, 1998), Izd. Mekh. Mat. Fak. Moskov. Univ., Moscow 1998, p. 43-49. (Russian)

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