THE NEIGHBORHOOD TOTAL EDGE DOMINATION NUMBER OF A JUMP GRAPH

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ABSTRACT

Let $J(G) = (V, E)$ be a jump graph without isolated vertices and isolated edges. An edge dominating set $F$ of $J(G)$ is called a neighborhood total edge dominating set if the edge induced sub graph $< N - F >$ has no isolated edges. The neighborhood total edge domination number $\nu_{nt}(J(G))$ of jump graph $J(G)$ is the minimum cardinality of neighborhood total edge dominating set of $J(G)$. In this paper, we initiate a study of this new parameter.

**Key words:** edge domination, connected edge domination, total edge domination, neighborhood total edge domination.

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1. INTRODUCTION:

All graphs considered here are finite, undirected without loops and multiple edges unless and otherwise stated, the graph $J(G) = (V, E)$ considered here have $p = |V|$ vertices and $q = |E|$ edges. Any undefined terms in this paper may found in Kulli [2].

A set $D$ of vertices in a graph $J(G)$ is called a dominating set if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\nu(J(G))$ of $J(G)$ is minimum cardinality of a dominating set of $J(G)$. {Anupama S.B. et al.,}

A set $E$ of edges in a jump graph $J(G)$ is called an edge dominating set if every edge in $E - F$ is adjacent to at least one edge in $F$. The edge domination number $\nu'(J(G))$ of $J(G)$. The concept of edge domination was introduced by Mitchell and Hedetniemi in [21] and was studied by several authors.

An edge dominating set $F$ of jump graph $J(G)$ is a connected edge dominating set if the edge induced sub graph $< F >$ is connected. The connected edge domination number $\nu'_c(J(G))$ of $J(G)$ is minimum cardinality of a connected edge dominating set of $J(G)$. The concept of connected edge domination was introduced by Kulli and Sigarkanti [16] and was studied [17]. A set $F$ of edges in a graph $J(G) = (V, E)$ is called total edge dominating set of $J(G)$ if every edge in $E$ is adjacent to at least one edge in $F$. The total edge domination number $\nu'_t(J(G))$ of $J(G)$ is the minimum cardinality of total edge dominating set of $J(G)$. This concept of edge domination number introduced by Kulli and Ptwari [15].

The vertices and edges of $J(G)$ are called elements of $J(G)$. A set $X$ of elements of $J(G)$ is an entire dominating set if every element not in $X$ is either adjacent or incident to at least one element in $X$. The entire domination number $\nu_e(J(G))$ of $J(G)$ is the minimum cardinality of entire dominating set of $J(G)$. A set $X$ of elements in $J(G)$ is a total entire dominating set if every element in a total entire dominating set if every element in $J(G)$ is either adjacent or incident to at least one element in $X$. The total entire domination number $\nu'_e(J(G))$ of $J(G)$ is the minimum cardinality of a total dominating set of $J(G)$. For any vertex $v \in V(J(G))$, the open neighborhood of $v$ the set $N(v) = \{ v \in V(J(G)) : uv \in E \}$ and closed neighborhood of $v$
v is the set N[v] = N(v) ∪ {v}. For a set S ∈ V(J(G)), he open neighborhood N(s) of S is defined by

N(S) = \bigcup_{v \in S} N(v) \quad \text{for all vertices } v \in S \text{ and closed neighborhood of } s \text{ is}

N[S] = N(S) \cup S. Let S be the set of vertices and let u ∈ S. The private neighbor set of u with respect to S is the set pn[u, S] = {v : N[S] \cap S = {u}}. For any edge e ∈ E the open neighborhood of e is N(e) and the closed neighborhood of e is N[e] = N(e) ∪ {e}. If F ⊆ E and e₁ = e₂ ∈ F, N(F) = ∪ N(e) and N[F] ∪ F if F ⊆ E and e ∈ S.

e₁ ∈ F, then private neighbor of e₁ with respect to F is the set

pn[e₁, F] = \{e₂ : N[e₂] \cap F = \{e₁\}\}. The degree of an edge uv is defined by deg u + deg v − 2. An edge uv is called an isolated edge if deg uv=0. Let Δ'(J(G)) denotes the maximum degree among the edges of J(G).

In the cycle C₉ = \{e₁, e₂, ..., e₉\}, F₁ = \{e₁, e₄, e₇\} and F₂ = \{e₂, e₄, e₆, e₈\} are edge dominating set of J(C₉). The induced sub graph < N(F₁)> has no isolated edges and the induced sub graph < N(F₂)> has isolated edges. We introduced the concept of neighborhood total edge domination number and study some parameters.

2. Results;
We assume throughout that J(G) in a jump gasph without isolated vertices and without isolated edges.

Definition 1. An edge dominating set F of a jump graph J(G) is called a neighborhood total edge dominating set if the induced sub graph < N – F > contains no isolated edges. The neighborhood total edge domination number √'ₚ(J(G)) of J(G) is the minimum cardinality of a neighborhood total edge dominating set of J(G).

Definition 2. A neighborhood total edge dominating set is minimal if no proper subset of F is a neighborhood total edge dominating set.

Proposition 3; For a jump graph J(G)

√'(J(G)) ≤ √'ₚ(J(G))..................(1)

Proof: Every neighborhood total edge dominating set is an edge dominating set. Thus (1) holds.

Theorem 4: If Pₙ is a path with n ≥ 4 vertices then √'(Pₙ) = \L_n/2. J.

Proof: let Pₙ = (V₁, V₂, .........Vₙ) be a path eith n ≥ 4 vertices. If n ≡ r (mod 4), r= 0 or 3 then F = \{eᵢ : i= 4k-2, 4k-1, k=1,2,....\} is a neighborhood total edge dominating set of Pₙ in n ≡ 2 (mod 4) then F ∪ \{eₙ-2\} is a neighborhood total edge dominating set of Pₙ.

Thus √'(Pₙ) = \L_n/2. J.

Further if n ≡ 2 (mod 4), then for any √'ᵢ-set F of Pₙ < N(F)> has at least one isolated edge. Thus √'ₚ(Pₙ) ≥ \L_n/2. J ≥ \L_n/2. J. Hence the result.

Corollary 5. If Pₙ is a path with n ≥ 4 vertices then √'ₚ(J(Pₙ)) = √'ᵢ(J(Pₙ)) if and only if n is even or n ≡ 1 (mod 4).

Proof: Since √'ᵢ(J(Pₙ)) = n/2 if n is even.
Theorem 6: If \( C_n \) is a cycle with \( n \geq 3 \) vertices then
\[
\sqrt{\frac{m}{3}}(J(C_n)) = \left\lfloor \frac{n}{3} \right\rfloor + 1 \quad \text{if } n \equiv 2 \pmod{3}
\]
\[
= \left\lfloor \frac{n}{3} \right\rfloor \quad \text{otherwise.}
\]

Proof: Let \( C_n = \{ v_1, v_2, \ldots, v_n, v_1 \} \) be a cycle with \( n \geq 3 \) vertices if \( n \equiv r \pmod{3} \), \( r = 0 \) or 1, then \( F = \{ e_i : i = 2k-2, k=1,2,\ldots \} \) is a neighborhood total edge dominating set of \( C_n \) if \( n \equiv 2 \pmod{3} \), then \( F \cup \{ e_n \} \) is a neighborhood total edge dominating set of \( C_n \).

Then
\[
\sqrt{\frac{m}{3}}(J(C_n)) = \left\lfloor \frac{n}{3} \right\rfloor + 1 \quad \text{if } p \equiv 2 \pmod{3}
\]
\[
= \left\lfloor \frac{n}{3} \right\rfloor \quad \text{otherwise}
\]

We have \( \sqrt{\frac{m}{3}}(J(C_n)) \geq \sqrt{\frac{m}{3}}(J(C_n)) = \left\lfloor \frac{n}{3} \right\rfloor \) if \( n \equiv 2 \pmod{3} \) then for any \( \sqrt{\frac{m}{3}} \)-set of \( F \) of \( J(C_n) \), \( \langle N(F) \rangle \) has at least one isolated edge. Thus \( \sqrt{\frac{m}{3}}(J(C_n)) \geq \left\lfloor \frac{n}{3} \right\rfloor + 1 \). Hence, the result.

Corollary 7: If \( C_n \) is a cycle with \( n \geq 3 \) vertices then
\[
\sqrt{\frac{m}{3}}(J(C_n)) = \sqrt{\frac{m}{3}}(J(C_n)) \quad \text{if and only if } n \equiv 0 \pmod{4}
\]
\[
\sqrt{\frac{m}{3}}(J(C_n)) = \sqrt{\frac{m}{3}}(J(C_n)) \quad \text{if and only if } n \equiv 2 \pmod{3}
\]
\[
\sqrt{\frac{m}{3}}(J(C_n)) = \sqrt{\frac{m}{3}}(J(C_n)) \quad \text{if and only if } n \equiv 3 \pmod{3}
\]
\[
\sqrt{\frac{m}{3}}(J(C_n)) = \sqrt{\frac{m}{3}}(J(C_n)) \quad \text{if and only if } n \equiv 1 \pmod{3}
\]

Proof: Since \( \sqrt{\frac{m}{3}}(J(C_n)) = \left\lfloor \frac{n}{2} \right\rfloor \) if \( n \equiv 0 \pmod{4} \)
\[
= \left\lfloor \frac{n}{2} \right\rfloor \quad \text{if } n \equiv 1 \pmod{4} \quad \text{or } n \equiv 3 \pmod{4}
\]
\[
= \left\lfloor \frac{n}{2} \right\rfloor + 1 \quad \text{if } n \equiv 2 \pmod{4}
\]

The result follows.

Theorem 8: If \( K_{m,n} \) is a complete bipartite graph with \( 2 \leq m \leq n \) then
\[
\sqrt{\frac{m}{3}}(J(K_{m,n})) = m
\]

Proof: In \( K_{m,n} \), \( v \) is a vertex such that \( \deg v = m \). Let \( F \) be the set of all edges incident with a vertex \( v \). It is easy to see that \( F \) is an edge dominating set and the induced sub graph \( \langle N(F) \rangle \) is connected and does not contain an isolated edge. Hence \( F \) is a neighborhood total edge dominating set. Then
\[
\sqrt{\frac{m}{3}}(J(K_{m,n})) \leq |F| = \deg v = m
\]
Since
\[
\sqrt{\frac{m}{3}}(J(K_{m,n})) = m
\]
The theorem follows.

Theorem 9: If \( K_p \) is a complete graph with \( p \geq 3 \) vertices then
\[
\sqrt{\frac{m}{3}}(J(K_p)) = \left\lfloor \frac{p}{2} \right\rfloor
\]

Proof: Let \( F \) be a minimum matching \( J(K_p) \). Clearly \( F \) is an edge dominating set and also \( \langle N(F) \rangle \) is connected and does not contain an isolated edge. Hence, \( F \) is a neighborhood total edge dominating set. Then
Theorem 10: A super set of a neighborhood edge total dominating set is a neighborhood total edge dominating set.

Proof; Let F be a neighborhood total edge dominating set of a jump graph J(G). Let $F_1 = F \cup \{e\}$ where $e \in E - F$. Then $e \in N(F_1)$ and $F_1$ is an edge dominating set of J(G). Suppose the induced subgraph $<N(F_1)>$ contains an isolated edge $e_1$. Then $N(e_1) \subseteq F - N(F)$. Then $e_1$ is an isolated edge in $<N(F)>$ which is a contradiction. Thus $<N(F_1)>$ has no isolated vertices. Therefore $F_1$ is a neighborhood total edge dominating set.

We establish a characterization of minimum neighborhood total edge dominating set.

Theorem 11: A neighborhood total edge dominating set $F$ of a jump graph $J(G)$ is minimal if and only if for every $e \in F$, one of the following holds

(i) $P_n[e,F]
eq \emptyset$

(ii) There exists an edge $e_1 \in N(F - \{e\})$ such that $N(e_1) \cap N(F - \{e\}) = \emptyset$

Proof: Let $F$ be a minimal neighborhood total edge dominating set of $J(G)$. Let $e \in F$. Then either $F - \{e\}$ is an edge dominating set. And the induced sub graph $<N(F - \{e\})>$ contains an isolated vertex. Suppose $F - \{e\}$ is not an edge dominating set. Then $P_n[e,F] = \emptyset$. Suppose $F - \{e\}$ is an edge dominating set and $e_1 = N(F - \{e\})$ is an isolated edge in $<N(F - \{e\})>$. Then $N(e_1) \cap N(F - \{e\}) = \emptyset$

Conversely consider $f$ a neighborhood total edge dominating set of $J(G)$ satisfying the conditions (i) and (ii). Then $F$ is a minimal neighborhood total edge dominating set. Thus by theorem 10 the result follows.

Theorem 12. Let $T$ be a tree. Then $\sqrt{n}(T) = 1$ if and only if $T = K_{1,p}$ or $p \geq 3$, or $S = m,n \geq 2$.

Proof: Let $T = P_3$ or $P_4$, then clearly $\sqrt{n}(J(T)) = 2$ Thus $T \neq P_3$ or $P_4$. Let $\sqrt{n}(J(T)) = 1$. Let $F = \{e\}$ be the $\sqrt{n}$-set of $J(T)$. Let $e = uv$, since $T \neq P_3$.

deg $v \geq 3$ suppose deg $u = 2$. Then $<N(F)>$ has two components in which one component is an isolated edge. Which is a contradiction. This implies that deg $u = 1$ or deg $u \geq 3$ If deg $u = 1$ then $\sqrt{n}(J(T)) = 1$ and $J(T) = K_{1,p}$.

If deg $u \geq 3$ and $J(T) = S = m,n \geq 2 \leq m \leq n$.

Converse is obvious.

Proposition 13; If $J(G)$ is a connected jump graph with $\Delta < q - 1$ then $\sqrt{n}(J(G)) = q - \Delta$.

Proof: Let $e$ be an edge of a connected jump graph $J(G)$ and deg $e = \Delta$. Since $\Delta < q - 1$, there exists two adjacency edge $e_1$ and $e_2$ such that $e_1 \neq e_2$, $e_1 \in N(e)$ and $e_2 \not\in N(e)$. Let $F = (N(e) - e_1) \cup \{e_2\}$. Then $|F| = \Delta$. Further it is easy to see that $E - f$ is a neighborhood total edge dominating set of $J(G)$. thus $\sqrt{n}(J(G)) \leq |E - F| = q - \Delta$.

Theorem 15: For any graph $J(G)$ $\sqrt{n}(J(G)) = q$ if and only if $J(G) = mP_3$.

Proof: Suppose $\sqrt{n}(J(G)) = q$ on the contrary assume $G \neq mP_3$. Then $J(G)$ has at least one component $G_1$ which is not $P_3$. Clearly all edges of $G_1$ are not in a neighborhood total edge dominating set. Hence $\sqrt{n}(J(G)) \neq q$ which is a contradiction. Hence $J(G) = mP_3$. 


REFERENCES