Connected Roman Semi Total Block Domination in Graphs

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ABSTRACT

A connected Roman dominating function $f = (V_0, V_1, V_2)$ on a semitotalblock graph $T_b(G) = H$ is a function $f: V(H) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$ such that $\langle V_1 \cup V_2 \rangle$ or $\langle V_2 \rangle$ is connected. The weight of a connected Roman dominating function is the value $f(V(H)) = \sum_{v \in V(H)} f(v)$. The minimum weight of a connected Roman dominating function on a semitotalblock graph $H$ is called the connected Roman semitotalblock domination number of $G$ and is denoted by $\gamma_{RCT}(G)$.

In this paper, we study the connected Roman domination number of semitotalblock graph $H$ and obtain some results on $\gamma_{RCT}(G)$ in terms of elements of $G$, but not in terms of $H$. Further we develop its relationship with other different domination parameters of $G$.

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INTRODUCTION

All graphs considered here are simple (finite, undirected and loop less). For standard graph theory the terminology not given here we refer [2] and [3]. Let $G = (V,E)$ be a graph with vertex set $V$ and edge set $E$, the open neighborhood $N(v)$ of the vertex $v$ consists of vertices adjacent to $v$ and the closed neighborhood of $v$ is $N[v] = v \cup N(v)$. For a subset $S \subseteq V(G)$, we define $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. The subgraph induced by $S$ is denoted by $\langle S \rangle$.

For any graph $G = (V,E)$, the semitotalblock graph $T_b(G) = H$ is the graph whose set of vertices is the union of the set of vertices and set of blocks of $G$ in which two vertices are adjacent if and only if the corresponding members of $G$ are adjacent or the corresponding members of $G$ are incident, see[6].

A set $D \subseteq V$ is a dominating set if every vertex not in $D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a minimal dominating set $D$ is called a domination number of $G$ and is denoted by $\gamma(G)$.

A dominating set $D$ is connected dominating set if an induced subgraph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ of a graph $G$ is the minimum cardinality of a connected dominating set. See[5].
A dominating set \( D \) is a total dominating set if the induced subgraph \( \langle D \rangle \) has no isolated vertices. The total domination number \( \gamma_t(G) \) of a graph \( G \) is the minimum cardinality of a total dominating set. See [5]

A Roman dominating function on a graph \( G = (V,E) \) is a function \( f : V \to \{0,1,2\} \) satisfying the condition that every vertex \( u \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \). The weight of a Roman dominating function is the value \( f(V) = \sum_{v \in V} f(v) \). The minimum weight of Roman dominating function on a graph \( G \) is called the Roman domination number of \( G \) and is denoted by \( \gamma_R(G) \). See[1]

A Roman dominating function \( f = (V_0, V_1, V_2) \) on a semitotal block graph \( T_b(G) = H \) is a connected Roman dominating function (CRDF) on \( G \) if \( \langle V_1 \cup V_2 \rangle \) or \( \langle V_2 \rangle \) is connected. The connected Roman semitotalblock domination number \( \gamma_{RC}(G) \) of \( G \) is the minimum weight of a connected Roman semitotalblock dominating function of \( G \).

We need the following theorems to prove our further results.

**Theorem A[8]:** For any graph \( G \), \( \gamma_k(G) \leq \gamma_{RC}(G) \).

**Theorem B[8]:** For any nontrivial tree \( T \), \( \gamma_{RC}(T) = 2\gamma(T) \) if and only if every non end vertex of \( T \) is adjacent to atleast one end vertex.

**Theorem C[4]:** If \( G \) is a connected graph with \( p \geq 3 \) vertices then \( \gamma_c(G) = p - k \) or \( \gamma_c(G) + k = p \) where \( k \) be the number of end vertices of \( T \).

**Theorem D[7]:** For any graph \( G \), \( \gamma_{RB}(G) \leq \gamma_R(G) \).

**Theorem E[9]:** Let \( G \) be any \((p,q)\) with \( p \geq 2 \) vertices then \( p \leq \gamma_{RS}(G) \).

**RESULTS**

In this section we illustrate the connected Roman semitotal block domination number by determining the value of \( \gamma_{RC}(G) \) for several classes of graphs.

**Theorem 1:**

1. For any path \( P_p \), \( p \geq 3 \) vertices,
   a. \( \gamma_{RC}(P_p) = p \), if \( p = 4, 5 \).
   b. \( \gamma_{RC}(P_{2k}) = 2k + n \), if \( p \neq 4, 5 \).
   c. \( \gamma_{RC}(P_{2k+1}) = 2k + n + 1 \).

2. For any graph \( G \), if \( G \) is a block or \( G \) has exactly one cut vertex incident with blocks which are complete, then \( \gamma_{RC}(G) = 2 \).

**Upper bounds for** \( \gamma_{RC}(G) \):

We establish the upper bounds for the connected Roman semitotalblock domination number.

**Theorem 2:** For any non trivial graph \( G \), \( \gamma_{RC}(G) \leq 2\gamma_c(G) \).

**Proof:** Let \( G \) be any non trivial connected graph with \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Suppose there exists a set \( D_c = \{v_1, v_2, \ldots, v_i\} \), \( 1 \leq i \leq n \) such that \( D_c \subset V(G) \) and \( \forall v_j \in V(G) - D_c \), \( 1 \leq j \leq n \) is adjacent to atleast one vertex of \( D_c \). Also there exists a unique path between every pair of vertices of \( D_c \). Then \( |D_c| = \gamma_c(G) \). Let the graph \( T_b(G) \) has \( \{b_i: 1 \leq i \leq n\} \) number of block vertices corresponding to the blocks \( B_i \in G; i = 1, 2, \ldots, n \). Then \( V[T_b(G)] = V(G) \cup \{b_i\} \). Suppose there exists the sets \( D_1, D_2 \) in \( T_b(G) \) such that \( D_1 \subseteq D_c \) and \( D_2 \subseteq \{b_i\} \). If \( f(V_1) = \phi \), then some
\( v_k \in D; 1 \leq k \leq i, \ v_m \in \{b_i\} \). Hence \( \{v_k \cup v_m\} \) forms a dominating set in \( T_b(G) \) in which \( \{v_k \cup v_m\} \in V_2 \) and if \( \{v_k \cup v_m\} \) is connected, then \( |v_k \cup v_m| = 2|D_c| \), which gives \( \gamma_{RCT}(G) = 2\gamma_c(G) \). If \( f(V_1) \neq \phi \), for some \( v_i \in V(G) \) or \( v_i \in D_c \) also belongs to \( V_1 \), then there exists \( v_k \subset v_i \) and \( v_m \subset v_i \) such that \( \{v_k \cup v_m\} \in V_2 \). If \( \{v_i \cup v_k \cup v_m\} \) is connected, then \( |v_i \cup v_k \cup v_m| \leq 2|D_c| \), which gives \( \gamma_{RCT}(G) \leq 2\gamma_c(G) \).

**Theorem 3:** For any graph \( G \) with \( n \) blocks such that each block is complete, then \( \gamma_{RCT}(G) \leq 2n \).

**Proof:** Let \( G \) be any graph with \( n \) blocks such that each block is complete and \( f = (V_0, V_1, V_2) \) be a \( \gamma_c \)-function in \( T_b(G) \). Let \( \{v_1, v_2, \ldots, v_n\} \) be the number of vertices incident with \( n \) blocks which forms a \( \gamma_c \)-set in \( G \). If every end blocks \( \{B_i; 1 \leq i \leq n\} \) of \( G \) contains the vertices \( \{v_i\} \subseteq \{v_n\} \), then \( \{v_i\} \in V_2 \). Otherwise there exists some non end blocks \( \{B_j; 1 \leq j \leq n\} \) contains the vertices \( \{v_j\} \subseteq \{v_n\} \), such that \( \{v_k, \{v_j\}\} \subseteq \{v_j\} \) and \( \{v_k\} \subseteq V_2 \), \( \{v_j\} \in V_1 \). If \( \{v_i\} \cup \{v_j\} \) is connected. Then \( ||v_i\cup v_j|| \leq 2||B_i\cup B_j|| = 2n \). Hence \( \gamma_{RCT}(G) \leq 2n \).

**Theorem 4:** Let \( G \) be any non trivial graph with \( G \neq P_p, p \geq 6 \) then \( \gamma_{RCT}(G) \leq p \).

**Proof:** Let \( G \) be any non trivial graph with \( G \neq P_p, p \geq 6 \). Suppose \( G = P_p, p \geq 6 \). Then by Theorem 1, \( \gamma_{RCT}(G) > p \), a contradiction. Hence \( G \neq P_p, p \geq 6 \). Suppose \( G = P_p \) with \( 1 \leq p \leq 5 \). Then by Theorem 1, the result is obvious.

Now let \( G \) be any non trivial graph with \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( f = (V_0, V_1, V_2) \) be a \( \gamma_c \)-function in \( T_b(G) \). Further \( D_c = \{v_1, v_2, \ldots, v_n\} \) \( 1 \leq i \leq n \) be the number of vertices which forms a \( \gamma_c \)-set of \( G \) and \( \{v_i\} \subseteq \{v_n\} \) be the number of vertices of \( G \) such that \( \forall v \in \{v_i\} \), \( G - v \) has atleast two components. Suppose every vertex of \( G \) \( \{v_j\} \) is adjacent to atleast one vertex of \( v_j \). Then \( \{v_j\} \subseteq V_1 \) or \( V_2 \) and \( G - \{v_j\} \subseteq V_0 \). Otherwise there exists atleast one vertex \( \{v_k\} \subseteq V - \{v_j\} \) which is not adjacent to \( \{v_j\} \). Let \( B_i; 1 \leq k \leq n \) be the number of blocks in \( G \) such that \( v_k \in \{B_k\} \). Then there exists the vertices \( \{b_k; 1 \leq k \leq n\} \) in \( T_b(G) \) corresponds to the blocks \( \{B_k\} \) of \( G \) such that \( \{b_k\} \subseteq V_2 \) and \( N(b_k) \cap \{v_1\} \subseteq V_1 \) or \( V_2 \) which gives \( |V_1| + 2|V_2| \leq |v_n| \). Hence \( \gamma_{RCT}(G) \leq p \).

**Theorem 5:** For any non trivial graph \( G \), \( \gamma_{RCT}(G) \leq \gamma_{RS}(G) \).

**Proof:** By Theorem 4 and Theorem E, \( \gamma_{RCT}(G) \leq p \leq \gamma_{RS}(G) \).

Hence \( \gamma_{RCT}(G) \leq \gamma_{RS}(G) \).

**Theorem 6:** For any non trivial graph \( G \), \( \gamma_{RCT}(G) \leq \gamma_c(G) + \gamma_i(G) \).

**Proof:** Let \( V(G) = \{v_1, v_2, \ldots, v_n\} \) be the number of vertices of \( G \). Let \( D_c = \{v_1, v_2, \ldots, v_n\} \) \( 1 \leq i \leq n \) be the number of vertices of \( G \) such that \( D_c \subset V \) and \( \{D_c\} \) is connected. Further \( D_i = \{v_1, v_2, \ldots, v_k\} \) \( 1 \leq k \leq n \) be the number of vertices of \( G \) and \( D_i \subset V \) such that \( \{D_i\} \) has no isolated vertices. Then \( |D_c| = \gamma_c(G) \) and \( |D_i| = \gamma_i(G) \). Let \( \gamma_c \)-set \( D_{RCT} \) such that \( |D_{RCT}| = \gamma_{RCT}(G) \). Since \( D_c \) is connected. Therefore the set \( D_c \) must contain atleast one vertex from each block \( \{B_i; 1 \leq i \leq n\} \). Then the block vertices \( \{b_i; 1 \leq i \leq n\} \) of \( T_b(G) \) corresponding to the blocks \( \{B_i; 1 \leq i \leq n\} \) are adjacent to atleast one vertex of \( D_c \). Suppose \( f(V_1) \neq \phi \) and \( f = (V_0, V_1, V_2) \) be a \( \gamma_{RC} \)-function in \( T_b(G) \) with \( \forall v_k \in V_1, 1 \leq k \leq n \). Then there exists atleast two vertices \( (u, w) \in N(v_k) \) such that \( (u \cup w) \in V_2 \). Also if there exist atleast
one vertex $v_m \subseteq D_c \cup D_i, 1 \leq m \leq n$. Suppose $v_m$ belongs to only one block $B$ of $G$. Then there exist a block vertex $b \in V[T_b(G)]$ is adjacent to at least one vertex of $D_c \cup D_i$ such that $b \in V_2$ in which $\{v_k \cup \{u \cup w\} \cup b\}$ forms a connected component. Then $|D_{RCT}| \leq |D_c| + |D_i|$. Suppose $f(V_1) = \emptyset$. Then $\forall v_i \in D_c \cup D_i, 1 \leq i \leq n, v_i \in V_2$, also $\{v_i\}$ forms a $\gamma_{RCT}$-set of $T_b(G)$ such that $\{v_i\}$ is connected, which gives $|D_{RCT}| \leq |D_c| + |D_i|$. Hence $\gamma_{RCT}(G) \leq \gamma_c(G) + \gamma_r(G)$.

**Theorem 7**: For any tree with $n$ blocks and $c \geq 1$ cut vertices, $\gamma_{RCT}(T) < n + c$.

**Proof**: Suppose $T$ be a tree with $n$ blocks and $A = \{v_1, v_2, ..., v_k\}$ be the number of cut vertices of $T$ such that $|A| = c$. Let $\{b_1, b_2, ..., b_n\}$ be the number of block vertices of $T_b(T)$ with respect to the blocks $\{B_1, B_2, ..., B_n\}$ of $T$ such that $|b_n| = n$. Since every block is $K_2$ in $T$ and each $v_k \in A; 1 \leq k \leq n$ is adjacent to its corresponding block vertex in $T_b(T)$ such that $v_k \in V_1$ or $V_2$, then the $\{V[T_b(T)] \circ A\}$ is not connected if it follows that $\{N(b_i) \cap A\}$ forms $\gamma_{RCT}$-set in $T_b(T)$ such that $\gamma_{RCT}(T) < n + c$. Therefore $\gamma_{RCT}(T) < n + c$.

Again as the number of blocks $B_n$ of $T$ increased by at least one block of $T$ then at least one cut vertex and one block vertex are increased in $T_b(T)$, which gives $\gamma_{RCT}(T) < n + c$.

The connected domination, Roman domination and connected Roman semitotalblock domination are related by the following inequality.

**Theorem 8**: For any graph $G, \gamma_{RCT}(G) \leq \gamma_c(G) + \gamma_r(G) - 1$.

**Proof**: Let $G$ be any graph and $f = (V_0, V_1, V_2)$ be a $\gamma_{RCT}$-function with connected Roman dominating set $D_{RCT}$ in $T_b(G)$ such that $|D_{RCT}| = \gamma_{RCT}(G)$. Now we consider the following cases.

**Case 1**: Suppose $G$ be any tree and $k$ be the number of end vertices in $G$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{RCT}$-function in $G$ and $f(V_1) = \emptyset$. Then $\forall u_i \in (V_2 \cap D_c)$ such that $u_i \in V_2$. Suppose there exist a vertex $w \in (V_0 \cap D_c)$. Then $w \in V_1$ or there exists the vertices $w_1, w_2 \in V_0 \cap D_c$. Clearly $w_1 \in V_1, w_2 \in V_2 \cup V_2$ or $w_1, w_2 \in V_1$. Suppose some $u_2 \in (V_2 \cap D_c)$. Then $u_2 \not\in V_2$ and $u_2 \in V_1$. Also there exists $u_2, u_2 \in V_0 \cap D_c$ where as $(u_2 \cup u_2) \in V_2$. Hence $\{V_1 \cup V_2\}$ forms a connected Roman dominating set in $T_b(G)$, which gives $|D_{RCT}| \leq |D_c| + |D_r| - 1$.

Thus $\gamma_{RCT}(G) \leq \gamma_c(G) + \gamma_r(G) - 1$.

**Case 2**: Suppose $G$ is not a tree. Then we consider the following subcases.

**Subcase 1**: Assume $G$ be a graph with $\{B_i; 1 \leq i \leq n\}$ blocks and $C$ be the number of cut vertices in $G$ such that every vertex of $B_i$ are adjacent to at least one cut vertex $c$ of $G$. Then $\forall v \in (V_2 \cap D_c), v \in V_2$ and if there exist a vertex $u \in V_0 \cap D_c$, then $u \in N(v)$ and hence $u \in V_1$. Also if there exists $w \in V_1 \cap D_c$ and $N(w) \in V_0 \cap D_c$. Then $w \in V_2$ and $N(w) \in V_1$, which gives $|D_{RCT}| \leq |D_c| + |D_r| - 1$.

Hence $\gamma_{RCT}(G) \leq \gamma_c(G) + \gamma_r(G) - 1$.
Subcase 2.2: Assume $G$ be a graph with $\{B_j: 1 \leq j \leq n\}$ blocks such that at least one vertex of each block is not adjacent to a cut vertex $c$. Then $\forall B_j$ in $G$ there exist corresponding block vertices $v_j \in b_j$ in $T_b(G)$ and hence $v_j \in V_2$ also $N(v_j) \cap D_c \subseteq V_1$ or $V_2$, which gives $|D_{RCT}| \leq |D_c| + |D_f| - 1$. Hence $\gamma_{RCT}(G) \leq \gamma_c(G) + \gamma_f(G) - 1$

In the following theorem, we establish the relation of $\gamma_{RCT}(G)$ with $\gamma_c(G)$ and blocks of $G$.

**Theorem 9:** For any graph $G$ with $n$ blocks $\gamma_{RCT}(G) \leq \gamma_c(G) + n$.

**Proof:** Let $G$ be any graph with $\{B_i: 1 \leq i \leq n\}$ be the number of blocks of $G$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{RC}$-function in $T_b(G)$.

Further $S = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$ be the set of all vertices with $\text{deg}(v) \geq 2$. Then there exists a minimal vertex set $S' \subseteq S$, which covers all the vertices of $G$. Clearly $S'$ forms a minimal $\gamma$-set of $G$. Suppose the subgraph $\langle S' \rangle$ has only one component. Then $S'$ itself is a connected dominating set of $G$. Otherwise, if the subgraph $\langle S' \rangle$ has more than one component, then attach the minimum number of vertices of $\{w_i\} \subseteq V(G) - S'$, where $\text{deg}(w_i) \geq 2$, which are between the vertices of $S'$ such that $s_i = S' \cup \{w_i\}$ forms exactly one component in the subgraph $\langle s_i \rangle$. Clearly, $s_i$ forms a minimal $\gamma_c$-set of $G$. Further $k$ be the number of end vertices of $G$. Then we consider the following cases.

**Case 1:** Suppose $k = \phi$. Then $k = 0 < n$. By Theorem C, $\gamma_c(G) + k = p$. Also by Theorem 4, $\gamma_{RCT}(G) \leq p$.

Hence $\gamma_{RCT}(G) \leq p = \gamma_c(G) + k < \gamma_c(G) + n$. Therefore $\gamma_{RCT}(G) \leq \gamma_c(G) + n$.

**Case 2:** Suppose $k \neq \phi$. Then consider the following subcases.

**Subcase 2.1:** Assume $G \neq K_2$. Then $\forall \{v_k\} \subseteq K_1 \leq k \leq n$ there exist the blocks $n \in B$, such that $k \leq n$. By Theorem C, $p = \gamma_c(G) + k \leq \gamma_c(G) + n$. Also by Theorem 4, $\gamma_{RCT}(G) \leq p$. Hence $\gamma_{RCT}(G) \leq \gamma_c(G) + n$.

**Subcase 2.2:** Assume $G = K_2$. Then by Theorem 1, $\gamma_{RCT}(G) = 2 = 1 + 1 = \gamma_c(G) + n$. Therefore $\gamma_{RCT}(G) \leq \gamma_c(G) + n$.

**Lower bound for $\gamma_{RCT}(G)$:**

In the following theorem, we obtain the lower bounds for $\gamma_{RCT}(T)$.

**Theorem 10:** For any non-trivial tree $T$, $\gamma(T) + c \leq \gamma_{RCT}(T)$, where $c$ be the number of cut vertices of $T$.

**Proof:** Suppose $G$ be a non-trivial tree with $V = \{v_1, v_2, \ldots, v_n\}$. Then there exists a set $V' \subseteq V$ such that $V' = \gamma(T)$. Now for any non-trivial tree $T$ with $B_j : 1 \leq i \leq n$ be the number of blocks of $T$. Then in $T_b(T)$, let $\{b_1, b_2, \ldots, b_n\}$ be the set of block vertices corresponding to $B_i$ in $T_b(T)$. We consider a function $f = (V_0, V_1, V_2)$ be a $\gamma_{RC}$-function in $T_b(T)$. Now $C_1 \subseteq C$ and $\forall \{v_i\} \subseteq C_1$. Then $\{v_i\} \subseteq V_2$. Otherwise $\forall \{v_i\} \subseteq C_1$ if $\{v_i\} \subseteq V$. Then $\{v_i\} \subseteq V$. It follows that the connected induced subgraph $\langle V_1 \cup V_2 \rangle$ or $\langle V_2 \rangle$ is $\gamma_{RCT}(T)$. Hence $\gamma(T) + c \leq \gamma_{RCT}(T)$.

In the following theorem, we obtain $\gamma_{RCT}(T)$ for which the lower bond is attained with connected Roman domination number of $T$.
Theorem 11: Let $T$ be any non trivial tree. Then $\gamma_{RC}(T) \leq \gamma_{RCT}(T)$.

Proof: Let $T$ be any non trivial tree and $f = (V_0, V_1, V_2)$ be a $\gamma_{RC}$-function in $T_k(T)$. Suppose $n_1 = \{v_1, v_2, \ldots, v_n\}$ be the number of non end vertices of $T$ adjacent to at least one end vertex and $n_2 = \{v_1, v_2, \ldots, v_i\}$ $1 \leq i \leq n$ be the number of end vertices of $T$ not adjacent to end vertices. Then we consider the following cases.

Case 1: Suppose $n_2 = \phi$. Then every non end vertex of $T$ is adjacent to at least one vertex. By Theorem 14, $\gamma_{RCT}(T) = 2\gamma(T)$. Also by Theorem B, $\gamma_{RC}(T) = 2\gamma(T)$. Hence $\gamma_{RC}(T) = \gamma_{RCT}(T)$.

Case 2: Suppose $n_2 \neq \phi$. Then $\forall \{v_i\} \in n_i ; i = 1, 2, \ldots, n$, $\{v_i\} \subseteq V_2$. Now consider $\{n_3, n_4\} \subseteq \{n_2\}$. Suppose $\forall u \in n_3, u \in V_2$. Then there exists at least one vertex $w \in n_4$ and $w \subseteq N(u)$ which gives $w \in V_1$. Suppose $u = \phi$. Then there exist $\{h_i; i = 1, 2, \ldots, n\} \in n_4$ such that $\{h_i\} \in V_1$. But $\{v_i\} \in V_2$ and $\{u \cup w\} \in V_1$, which gives $\gamma_{RC}(T) < \gamma_{RCT}(T)$. Hence $\gamma_{RC}(T) \leq \gamma_{RCT}(T)$.

Theorem 12: For any non trivial tree $T$, then $\gamma_r(T) \leq \gamma_{RCT}(T)$.

Proof: From Theorem A and Theorem 11, $\gamma_r(T) \leq \gamma_{RC}(T) \leq \gamma_{RCT}(T)$. Hence $\gamma_r(T) \leq \gamma_{RCT}(T)$.

Theorem 13: For any tree $T$, then $\gamma_{RB}(T) \leq \gamma_{RCT}(T)$.

Proof: By Theorem D, $\gamma_{RB}(T) \leq \gamma_r(T)$ and by Theorem 12, $\gamma_r(T) \leq \gamma_{RCT}(T)$. Hence $\gamma_{RB}(T) \leq \gamma_{RCT}(T)$.

In the following theorem we establish the equality result.

Theorem 14: For any non trivial tree $T$, if every non end vertex of $T$ is adjacent to at least one end vertex. Then $\gamma_{RCT}(T) = 2\gamma(T)$.

Proof: Let $T$ be any non trivial tree. Further $f = (V_0, V_1, V_2)$ be a $\gamma_{RC}$-function with connected Roman dominating set $D_{RCT}$ in $T_k(T)$ such that $|D_{RCT}| = \gamma_{RCT}(T)$ and $k$ be the number of end vertices of $T$. Suppose $n_1 = \{v_1, v_2, \ldots, v_n\}$ be the number of non end vertices of $T$ adjacent to at least one end vertex. Then there exists a minimal $\gamma$-set $D$ of $T$ such that $|D| = \gamma(T) = n_1$. Let $\{b_i; 1 \leq i \leq n\}$ be the number of blocks of $T$ corresponding to the block vertices $\{b_i; 1 \leq i \leq n\}$ of $T_k(T)$. Since each $v \in n_i$ is incident with at least two blocks and is adjacent to $\{k \cup b_i\}$ in $T_k(T)$ with the property $|k \cup b_i| = |V_0|$, $|n_i| = |V_2|$ and $|V_1| = \phi$, which gives $|D_{RCT}| = |V_0| + 2|V_2| = \phi + 2n_1 = 2|D|$. Hence $\gamma_{RCT}(T) = 2\gamma(T)$.

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