A Study Of Induced Fuzzy Topology

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ABSTRACT
The aim of the present paper is to introduce the concept of induced fuzzy topology and have verified that the induced topology τ_A *is actually a fuzzy topology on A.*

Key words : *induced fuzzy topology, fuzzy topological space, fuzzy set, subspace.*

INTRODUCTION

The concept of induced fuzzy topological space, introduced by Weiss (1975), was defined with the notions of a lower semi-continuous function. In 1976, R. Lowen noticed the natural association between a topological space and a fuzzy topological space on a set X and introduced the notion of topologically generated space which is same as the induced fuzzy topological space of Weiss and further studied category of the fuzzy topological space. Since then several other authors continued the investigation of such spaces.

Let (X, τ) be a fuzzy topological space and A be a fuzzy set in X. Let $\tau_A$be collection of those fuzzy sets which are intersection of A with members of τ. i.e.

$$\tau_A = (A \cap G : G \in \tau\}$$

Then τ_A is a topology for A, called the induced fuzzy topology and the fuzzy topological space (A, τ_A) is called a subspace of (X, τ). Every member of τ_A is called relatively open in A.

Now we shall show that (A, τ_A) is a fuzzy topological space, for any $G \in \tau$, $A \cap G$ is a fuzzy set, of which the membership function is defined by

$$\mu_{A \cap G}(x) = \min\{\mu_A(x), \mu_G(x)\}; \qquad \forall x \in X$$

where τ_A and τ_G are membership functions of A and G respectively. Therefore it is obvious that

$$\mu_{A \cap G}(x) \leq_A(x); \qquad \forall x \in X$$

Thus $\qquad A \cap G \subseteq A$

Since G is an arbitrary, element of τ, each fuzzy set which is the intersection of A with a number of τ is a fuzzy subset of A.

Therefore τ_A is a collection of fuzzy sets in A.

Now since $X \in \tau$; $A \cap X \in \tau_A$; what the membership relation of X is defined by

$$\mu_x(x) = 1; \quad \forall x \in X$$

Now we have

$$\mu_{A \cap x}(x) = \min\{\mu_A(x), \mu_x(x)\}; \qquad \forall x \in X$$
$$= \min\{\mu_A(x), 1\}; \qquad \forall x \in X \quad [\because \mu_x(x) = 1]$$
$$= \mu_x(x); \qquad \forall x \in X$$

[since the membership grade of any element in any fuzzy set ≤ 1]

Thus $\mu_{A \cap x}(x) = \mu_A$

Therefore $A \cap X = A$

Hence as $A \cap X \in \tau_A$; $\qquad A \in \tau_A$

Again since $\phi \in \tau$, $A \cap \phi \in \tau_A$ and the membership function of ϕ is defined by

$$\mu_\phi(x) = 0; \quad \forall x \in X$$

Now, $\mu_{A \cap \phi}(x) = \min\{\mu_A(x), \mu_\phi(x)\}; \qquad \forall x \in X$
$= \min\{\mu_A(x), 0\}; \qquad \forall x \in X$
$= 0 \qquad$ [Since the membership grade of any element in any fuzzy ≥0]
$= \mu_x(x); \qquad \forall x \in X$

So, $\qquad \mu_{A \cap \phi}(x) = \mu_\phi(x); \qquad \forall x \in X$

$$\Rightarrow \mu_{A \cap \phi} = \mu_\phi$$

Therefore $A \cap \phi = \phi$

As $A \cap \phi \in \tau_A$; $\quad \phi \in \tau_A$

Now, Suppose that $A \cap G \in \tau_A$and$A \cap H \in \tau_A$; where μ_G and $\mu_H$are the membership functions of G and Hrespectively.

Therefore by definition $G \in \tau$ and $H \in \tau$. For $x \in X$, we have

$$\mu_{(A\cap G)\cup(A\cap H)}(x) = \max\{\mu_{(A\cap G)}(x), \mu_{(A\cap H)}(x)\}$$
$$= \max[\min\{\mu_A(x), \mu_G(x), \min\{\mu_A(x), \mu_H(x)\}]$$

Now, four cases arise

Case I : When $\mu_A(x) \le \mu_G(x)$ and $\mu_A(x) \le \mu_H(x)$

Then $\mu_{(A\cap G)\cup(A\cap H)}(x) = \max\{\mu_A(x), \mu_A(x)\}$

$$= \mu_A(x) = \min[\{\mu_A(x), \max\{\mu_G(x), \mu_H(x)\}]$$
$$= \min[\{\mu_A(x), \mu_{G\cup H}(X)\}] = \mu_{A\cap(G\cup H)}(x)$$

Case II : When $\mu_A(x) \le \mu_G(x)$ and $\mu_H(x) \le \mu_A(x)$

Thus $\mu_H(x) \le \mu_A(x) \le \mu_G(x)$

Then $\mu_{(A\cap G)\cup(A\cap H)}(x) = \max\{\mu_A(x), \mu_H(x)\}$

$$= \mu_A(x) = \min\{\mu_A(x), \mu_G(x)\} = \min[\{\mu_A(x), \max\{\mu_G(x), \mu_H(x)\}]$$
$$= \min[\{\mu_A(x), \mu_{G\cup H}(X)\}] = \mu_{A\cap(G\cup H)}(x)$$

Case III : When $\mu_A(x) \le \mu_H(x)$ and $\mu_G(x) \le \mu_A(x)$

Thus $\mu_H(A) \le \mu_A(x) \le \mu_H(x)$

Then $\mu_{(A\cap G)\cup(A\cap H)}(x) = \max\{\mu_G(x), \mu_A(x)\}$

$$= \mu_A(x) = \min\{\mu_A(x), \mu_H(x)\} = \min[\{\mu_A(x), \max\{\mu_G(x), \mu_H(x)\}]$$
$$= \min[\{\mu_A(x), \mu_{G\cup H}(x)\}] = \mu_{A\cap(G\cup H)}(x)$$

Case IV : When $\mu_G(x) \le \mu_A(x), \mu_H(x) \le \mu_A(x)$

Then $\mu_{(A\cap G)\cup(A\cap H)}(x) = \max\{\mu_G(x), \mu_H(x)\}$

$$= \min[\{\mu_A(x), \max\{\mu_G(x), \mu_H(x)\}]$$
$$= \min[\{\mu_A(x), \mu_{G\cup H}(x)\}]$$
$$= \mu_{A\cap(G\cup H)}(x)$$

Thus is each ease we get

$$\mu_{(A\cap G)\cup(A\cap H)}(x) = \mu_{A\cap(G\cup H)}(x), \qquad \forall x \in X$$
$$\Rightarrow \mu_{(A\cap G)\cup(A\cap H)} = \mu_{A\cap(G\cup H)}$$
$$\Rightarrow (A\cap G)\cup(A\cap H) = A\cap(G\cup H)$$

Now since $G \in \tau$ and $H \in \tau$; $G\cup H \in \tau$ ($\because \tau$ is a fuzzy topology)

Thus $A\cap(G\cup H) \in \tau_A$

Therefore $(A\cap G)\cup(A\cap H) \in \tau_A$

Hence for any $A\cap G \in \tau_A$ and $A\cap H \in \tau_A$ we have $(A\cap G)\cup(A\cap H) \in \tau_A$

Finally we shall show that $(A\cap G)\cap(A\cap H) \in \tau_A$ for any $A\cap G \in \tau_A$ and $A\cap H \in \tau_A$

Now we have for any $x \in X$

$$\mu_{(A\cap G)\cap(A\cap H)}(x) = \min\{\mu_{(A\cap G)}(x), \mu_{(A\cap H)}(x)\}$$
$$= \min[\min\{\mu_A(x), \mu_G(x), \min\{\mu_A(x), \mu_H(x)\}]$$
$$= \min[\{\mu_A(x), \min \mu_G(x), \mu_H(x)\}]$$
$$= \min[\{\mu_A(x), \mu_{G\cap H}(x)\}] = \mu_{A\cap(G\cap H)}(x)$$

Thus $\mu_{(A\cap G)\cap(A\cap H)} = \mu_{A\cap(G\cap H)}$

Therefore $(A\cap G)\cap(A\cap H) = A\cap(G\cap H)$

Since $G \in \tau, H \in \tau$ and $G\cap H \in \tau$

So A $A\cap(G\cap H) \in \tau_A$

Thus $(A\cap G)\cap(A\cap H) \in \tau_A$; for any $A\cap G \in \tau_A, A\cap H \in \tau_A$

Therefore τ_A is a fuzzy topology for A, consequently (A, τ_A) is a fuzzy topological space.

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