SOLUTION OF DIFFUSION EQUATION WITH CONSTANT CO-EFFICIENT IN CYLINDRICAL AND SPHERICAL COORDINATES

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Abstract: This paper aims to apply the variables separation Method to solve the three-dimensional Diffusion equation with constant coefficient in cylindrical and spherical coordinates. Illustrative some examples are related to known results.

Keywords: cylindrical coordinates, spherical coordinates

Basic definitions:

The diffusion equation in one dimensional:

\[
\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}.
\]

On the interval \( x \in [0, L] \) with initial condition

\[ u(x, 0) = f(x), \quad \forall x \in [0, L] \]

And dirichlet boundary condition

\[ u(0, t) = u(L, t) = 0 \quad \forall t > 0 \]

The diffusion equation in two dimensional:

\[
\frac{\partial u}{\partial t} = D \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

Where \( u = u(x, y, t), x \in [a_x, b_x], y \in [a_y, b_y], \) the second-order derivative in space leads to a demand for two boundary conditions.

The diffusion equation in three dimensional:

\[
\frac{\partial T}{\partial t} = \alpha \nabla^2 T
\]

Where \( T \) is a temperature and \( \alpha \) is a diffusion coefficient.

Bessel differential equation:

The Bessel differential equation is the linear second-order ordinary differential equation given by

\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2)y = 0
\]
SOLUTION OF DIFFUSION EQUATION IN CYLINDRICAL CO-ORDINATES

Consider a three-dimensional diffusion equation:
\[
\frac{\partial T}{\partial t} = \alpha \nabla^2 T
\]

Where \( T \) is a temperature and \( \alpha \) is a diffusion coefficient

In cylindrical co-ordinates \((r,\theta,z)\) the above equation becomes
\[
\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}
\]
(1.1)

Where \( T=T(r,\theta,z,t) \)

Let us assume separation of variables in the form \( T(r,\theta,z,t) = R(r)H(\theta)Z(z) \beta(\alpha,t) \)

Substitute in equation (1.1), it becomes
\[
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{H'}{H} + \frac{Z''}{Z} = \frac{1}{\alpha} \frac{\beta'}{\beta} = \lambda^2 \text{ (say)}
\]
(1.2)

Where \( \lambda^2 \) is a separation constant.

Dividing by \( R \) \( H \) \( \beta \)

\[
\beta' + \alpha \lambda^2 \beta = 0
\]
(1.2)

\[
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{H'}{H} + \lambda^2 = \frac{Z''}{Z} = -\mu^2 \text{ (say)}
\]
(1.3)

where \( \mu^2 \) is a separation constant.

Thus the equation determining \( Z \), \( R \) and \( H \) becomes
\[
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{H'}{H} + \lambda^2 + \mu^2 = 0
\]
(or)
\[
r^2 \frac{R''}{R} + r \frac{R'}{R} + (\lambda^2 + \mu^2)r^2 = -\frac{H''}{H} = \nu^2 \text{ (say)}
\]

Therefore,
\[
H'' + \nu^2 H = 0
\]
(1.4)

\[
R'' + \frac{1}{r} \frac{R'}{r} + [(\lambda^2 + \mu^2) - \frac{\nu^2}{r^2}]R = 0
\]
(1.5)

Equation (1.2)-(1.4) have particular solutions of the form
\[
\beta = e^{-\alpha \lambda^2 t}
\]
H = C cos νθ + sin νθ
Z = A e^{μz} + B e^{-μz}

The diffusion equation (1.5) \( R'' + \frac{1}{r} R' + \left[ (\lambda^2 + \mu^2 - \frac{v^2}{r^2}) \right] R = 0 \) is called Bessel’s equation of order \( v \) and its general solution is

\[ R(r) = C_1 J_v(\sqrt{(\lambda^2 + \mu^2) r}) + C_2 Y_v(\sqrt{(\lambda^2 + \mu^2) r}) \]

Where \( J_v(r) \) and \( Y_v(r) \) are Bessel functions of order \( v \) of the first and second kind, respectively.

Equation (1.5) is singular when \( r=0 \). The physically meaningful solutions must be twice continuously differentiable in \( 0 \leq r \leq a \).

Hence, Equation (1.5) has only one bounded solution,

\[ R(r) = J_v(\sqrt{(\lambda^2 + \mu^2) r}) \]

Finally, the general solution of the equation (1.1) is given by

\[ T(r, \theta, z, t) = e^{-\alpha \lambda^2 t} \left[ A e^{\mu z} + B e^{-\mu z} \right] \left[ C \cos \nu \theta + D \sin \nu \theta \right] J_v(\sqrt{(\lambda^2 + \mu^2) r}) \]

**EXAMPLE 1**

Determine the temperature \( T(r, t) \) in the infinite cylinder \( 0 \leq r \leq a \) when the initial temperature is \( T(r, 0) = f(r) \) and the surface \( r=a \) is maintained at 0° temperature.

Solution:

The governing PDE from the data of the problem is

\[ \frac{\partial T}{\partial t} = \alpha \nabla^2 T \]  \hspace{1cm} (1.6)

Where \( T \) is a function of \( r \) and \( t \) only. Therefore

\[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]

The corresponding boundary and initial conditions are given by

Boundary condition: \( T(a, t) = 0 \)

Initial condition: \( T(r, 0) = f(r) \)

The general solution of equation (1.6) is

\[ T(r, t) = A \exp(-\alpha \lambda^2 t) J_0(\lambda r) \]

Using the boundary condition, we obtain

\[ J_0(\lambda a) = 0 \]

Which has an infinite no. of roots, \( \lambda_n \) (\( n = 1, 2, 3, \ldots \infty \)). Thus, we get from the superposition principle the equation is

\[ T(r, t) = \sum_{n=1}^{\infty} A_n \exp(-\alpha \lambda_n^2 t) J_0(\lambda_n r) \]

Using initial condition \( T(r, 0) = f(r) \) we get,

\[ f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \]
to compute $A_n$ multiply both sides by $r J_0(\epsilon_m r)$ and integrate with respect to $r$

$$
\int_0^a r f(r) J_0(\epsilon_m r) dr = \sum_{n=1}^\infty A_n \int_0^a r J_0(\epsilon_m r) J_0(\epsilon_n r) dr
$$

$$
= \begin{cases} 
0 & \text{for } n \neq m \\
A_m \left( \frac{\alpha^2}{2} \right) a J_1^2(\epsilon_m a) & \text{for } n = m
\end{cases}
$$

Which gives

$$
A_m = \frac{2}{a^2 J_1^2(\epsilon_m a)} \int_0^a u f(u) J_0(\epsilon_m u) du
$$

Hence the final solution of the problem is

$$
T(r,t) = \frac{2}{a^2} \sum_{m=1}^\infty \frac{J_0(\epsilon_m r)}{J_1(\epsilon_m a)} \exp(-\alpha \epsilon_m^2 t) \left[ \int_0^a u f(u) J_0(\epsilon_m u) du \right]
$$

**SOLUTION OF DIFFUSION EQUATION IN SPHERICAL COORDINATES**

Consider a 3-D diffusion equation,

$$
\frac{\partial T}{\partial t} = \alpha \nabla^2 T
$$

Let $T = T(r,\theta,\phi,t)$

This equation can be written in spherical coordinates,

$$
\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 T}{\partial \phi^2} = \frac{1}{a \alpha} \frac{\partial T}{\partial t}
$$

This equation is separated by assuming the temperature function of the form

$$
T = R(r) H(\theta) \Phi(\phi) \beta(t)
$$

Substituting (2.2) in (2.1), we get

$$
\frac{R''}{R} + \frac{2 R'}{R} + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dH}{d\theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{d^2 \Phi}{d\phi^2} = \frac{1}{a \alpha} \frac{\beta'}{\beta} = -\lambda^2 \text{(Say)}
$$

Where $\lambda^2$ is a separation constant. Thus,

$$
\frac{d\beta}{dt} - \lambda^2 \alpha \beta = 0
$$

$$
\int \frac{d\beta}{dt} = \lambda^2 \alpha \int dt
$$

$e^{\lambda^2 \alpha t} = e^{-\lambda^2 \alpha t}$
\[
\beta = C_1 e^{-\lambda^2 ct} \quad (2.3)
\]

Also,
\[
r^2 \sin^2(\theta) \left[ \frac{1}{r} \frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} \right] \left[ \frac{1}{r^2 \sin \theta} H \frac{dH}{d\theta} \right] \left[ \frac{1}{r^2 \sin \theta} H \frac{dH}{d\theta} + \lambda^2 \right] = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = m^2 \text{ (say)}
\]

Which gives
\[
\frac{d^2 \Phi}{d\phi^2} + m^2 \Phi = 0
\]

Whose solution is
\[
\Phi(\phi) = C_1 e^{im\phi} + C_2 e^{-im\phi} \quad (2.4)
\]

The other separated equation is
\[
\frac{r^2}{R} \left( R'' + \frac{2}{r} R' + \lambda^2 r^2 \right) = m^2 \frac{r^2}{\sin^2(\theta)} \sin(\theta) \frac{d^2 H}{d\theta^2} + \frac{1}{H} \sin \theta \frac{dH}{d\theta} = n(n+1) \text{ (say)}
\]

On re-arrangement, this equation can be written as
\[
R'' + \frac{2}{r} R' + \left( \lambda^2 - \frac{n(n+1)}{r^2} \right) R = 0
\]

\[
-\frac{1}{H} \sin(\theta) \sin \theta \frac{d^2 H}{d\phi^2} + \cos \theta \frac{dH}{d\phi} + \frac{m^2}{\sin^2(\theta)} n(n+1) H = 0 \quad (2.6)
\]

Let \( r = (\lambda r)^{-1/2} \psi(r) \) then Eq(2.6) becomes
\[
(\lambda r)^{-1/2} \left[ \psi'' + \frac{1}{r} \psi'(r) + \left( \lambda^2 - \frac{(n+1/2)^2}{r^2} \right) \psi \right] = 0 \quad \text{since } (\lambda r) \neq 0
\]

We have
\[
\psi''(r) + \frac{1}{r} \psi'(r) + \left( \lambda^2 - \frac{(n+1/2)^2}{r^2} \right) \psi(r) = 0
\]

The above equation is the Bessel’s equation of order (n+1/2).
whose solution is

\[ \Psi(r) = A J_{n+1/2}(\lambda r) + B Y_{n+1/2}(\lambda r) \]

\[ R(r) = (\lambda r)^{-1/2} [A J_{n+1/2}(\lambda r) + B Y_{n+1/2}(\lambda r)] \] (2.7)

Where \( J_n \) and \( Y_n \) are Bessel’s function of first and second kind respectively.

Now equation (2.7) can be put in a more convenient form by introducing a new independent variable

\[ \mu = \cos \theta \quad (\cot \theta = \mu / \sqrt{1 - \mu^2}) \]

\[ (1 - \mu^2) \frac{d^2H}{d\mu^2} - 2 \mu \frac{dH}{d\mu} + \left[ n(n+1) - \frac{m^2}{1 - \mu^2} \right] H = 0 \]

Thus (2.6) equation becomes

Which is an associated legendra differential equation. Whose solution is

\[ H(\theta) = A_1 P_n^m(\mu) + A_2 Q_n^m(\mu) \]

Where \( P_n^m(\mu) \) and \( Q_n^m(\mu) \) are legendra function of degree \( n \) order \( m \), of first and second kind, respectively.

The physically meaningful general solution of the diffusion equation in spherical geometry is of the form

\[ T(r, \theta, \phi, t) = \sum_{\lambda, m, n} A_{\lambda mn}(\lambda r)^{-1/2} J_{n+1/2}(\lambda r) P_n^m(\cos \theta) e^{\pm \imath m \phi - \alpha \lambda^2 t} \]

In this general solution, the function \( Q_n^m(\mu) \) and \( (\lambda r)^{-1/2} Y_{n+1/2}(\lambda r) \) are excluded because these function have poles at \( \mu = \pm 1 \) and \( r = 0 \) respectively.

**Example 2**

Find the temperature in a sphere of radius \( a \) when its surface is kept at 0 temperature and its initial temperature is \( f(r, \theta) \).

**Solution:**

Here, the temperature is governed by the 3-D heat equation in spherical polar coordinates independent of therefore, the task is to find the solution of PDE.

\[ \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{\alpha} \frac{\partial T}{\partial t} = 0 \] (2.8)

Subject to
Boundary condition \( T(a, \theta, t) = 0 \) \quad (2.9)

Initial condition \( T(r, \theta, 0) = f(r, 0) \) \quad (2.10)

The general solution of equation (2.8) with the help of Eq(2.9), can be written as

\[
T(r, \theta, t) = \sum_{\lambda, n} A_{\lambda n}(\lambda r)^{-\frac{1}{2}} J_{n+1/2}(\lambda r) P_n(\cos \theta) e^{-\alpha \lambda^2 t} \quad (2.11)
\]

Applying the boundary condition we get,

\[ J_{n+1/2}(\lambda a) = 0 \]

This equation has infinitely many positive roots. Denoting them by \( \epsilon_j \), we have

\[
T(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} A_{n j}(\epsilon_j r)^{-1/2} J_{n+1/2}(\epsilon_j r) P_n(\cos \theta) \exp(-\alpha \epsilon_j^2 r) \quad (2.12)
\]

Now applying the initial condition and denote \( \cos \theta \) by \( \mu \), we get

\[
f(r, \cos^{-1}(\mu)) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} A_{n j}(\epsilon_j r)^{-1/2} J_{n+1/2}(\epsilon_j r) P_n(\mu)
\]

Multiply both sides by \( P_m(\mu) \) and integrating between the limits, -1 to 1, we obtain

\[
\int_{-1}^{1} f(r, \cos^{-1}(\mu)) P_m(\mu) d\mu = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} A_{n j}(\epsilon_j r)^{-1/2} J_{n+1/2}(\epsilon_j r) \int_{-1}^{1} P_m(\mu) P_n(\mu) d\mu
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} A_{n j} (\epsilon_j r)^{-1/2} J_{n+1/2}(\epsilon_j r) \left( \frac{2}{2n+1} \right)
\]

(or)

\[
\left( \frac{2}{2n+1} \right) \int_{-1}^{1} f(r, \cos^{-1}(\mu)) P_m(\mu) d\mu = \sum_{j=1}^{\infty} A_{n j} (\epsilon_j r)^{-1/2} J_{n+1/2}(\epsilon_j r) \quad \text{for } n=0,1,2,\ldots
\]

Now, to evaluate the constant \( A_{n j} \)

Multiply both side of the above equation by \( \epsilon_j^2 J_{n+1/2}(\epsilon_j r) \) and integrate with respect to \( r \) get
Thus, equations (2.12) and (2.13) together constitute the solution for the given problem.

Conclusion

The expectation of using variables separable method and obtaining better results, in a very expressive way was achieved.