HISTORICAL AND RECENT DEVELOPMENTS IN FRACTIONAL CALCULUS: A SURVEY

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Abstract: The aim of this paper, to provide the detail survey of historical and recent developments of fractional calculus. Recently fractional calculus has been attracted much attention since it plays an important role in many fields of science and engineering.

Index Terms - Fractional Calculus, Historical and recent developments, Fractional operators, Fractional differential equations.

I. INTRODUCTION

Now days, the fractional calculus is new field of mathematical study that deals with the investigation and applications of derivatives and integrals of non-integer order. Fractional calculus has been 300 years old history, the development of fractional calculus is mainly focused on the pure mathematical field. The earliest systematic studies seem to have been made in the 19^{th} century by Liouville, Riemann, Leibniz etc. In the last two decades fractional differential equations (FDEs) have been used to model various stable physical phenomena with anomalous decay. Many mathematical models of real problems arising in various fields of science and engineering are either linear systems or non-linear systems. Now the development of fractional calculus, it has been found that the behavior of many systems can be described by using the differential systems. It is worth mentioning that may physical phenomena having memory and genetic characteristics can be described by using the fractional differential systems. In fact, most of the real world processes are fractional order systems. That means a lot of physical systems show fractional dynamical behavior because of special materials and chemical properties. Fractional differential equations arise as a mathematical modeling of systems and processes in the field of physics, chemistry, biology, economics, control theory, signal and image processing, bio-physics, polymer rheology, aerodynamics etc. the model problems is very difficult to handle and to obtain its analytical solutions. Also, sometimes it is very difficult to solve these modeled problems because of its non linearity and complex geometry. Recently, researchers developed some fractional order finite difference schemes and iterative methods for fractional differential equations and obtain its solution [10,31]. Therefore, there are some finite difference methods (FDM's) and decomposition methods exist to solve such modeled problems.

HISTORICAL DEVELOPMENT:

In a letter, dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function $f(x) = x$, $\frac{D^n x}{Dx^n}$. L'Hopital's posed the question to Leibniz, what will be the result if $n = \frac{1}{2}$? Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn"[19]. Therefore, on that date fractional calculus was born and hence it is the birthday of fractional calculus. Furthermore, many welknown mathematicians like Fourier, Euler and Laplace etc. contributed in the development of fractional calculus. Also, many mathematicians used their own notations and defined the concept of non-integer order integral and derivatives. In the 20^{th} century, most of the mathematical theory of fractional calculus is developed. However it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found. The most famous of these definitions that have been popularized in the world of fractional calculus (not yet the world as a whole) are the Riemann-Liouville and Grunwald-Letnikov definition. Furthermore, Caputo reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [25]. In 1996, Kolowankar reformulated again, the Riemann-Liouville fractional derivative in order to differentiate no-where differentiable fractal functions [15]. The remarkable work began only in the nineteenth century with many great mathematicians like Euler, P. S. Laplace (1812), J. B. J. Fourier (1822), N.H. Abel (1823−1826), J. Liouville (1832−1873), B. Riemann (1847), H.Holmgren (1865−67), A. K. Grunwald (1867−1872), A. V. Letnikov (1868−1872), H. Laurent (1884), P. A. Nekrassov (1888) etc. and contributing to its development. However, recent attempt is on to have definition of fractional derivative as local operator specifically to fractal science theory. Perhaps, *Fractional Calculus is the calculus of twenty first century*. Heaviside (1922), Buss (1929), Goldman (1949), Starkey (1954), Holbrook (1966), Oldham and Spanier (1974), L. Debnath (1992), Miller and Ross (1993), R. K. Saxena (2002), Igor Podlubny (2003), S.Saha Ray (2005), and several others devoted additionally very reliable contributions to the subject. N. Ya. Sonin, (1869) A. Krug (1890), J. Hadmard (1892), S. Pinchorle (1902), G.H.Hardy and J.E.Littlewood (1917-28), H. Weyl (1917), P.Levy (1923), A. Marchaud (1927), H.T.Davis (1924-36), E.L.Post (1930), A.Zygmund (1935-45), E.R.Love (1938-96) A.Erdelyi (1939-65), H.Kober (1940), D.V.Wider (1941), M.Riesz (1949), W. Feller (1952), K. Nishimoto (1987), Caputo (1967).

In recent years, fractional differential equations have been investigated by many authors. Rawashdeh used the collection spline method to approximate the solution of fractional equations.

Momani obtained local and global existence and uniqueness solution of the integro-differential equation. Yong Zhou was devoted to a rapidly developing area of the research for the qualitative theory of fractional differential equations. In particular, he was interested in the bvasic theory of fractional differential equations. Such basic theory should be the starting point for further research concerning the dynamics, control, numerical analysis and applications of fractional differential equations. He presents some techniques for the investigation of fractional evolution equations governed by C_0 semigroup [33]. He also doing the work on recent advances on theory for fractional partial differential equations including fractional Euler-Lagrange equations, time-fractional diffusion equations, fractional Hamiltonian systems and and fractional Schrodinger equations. N.Ya.Sonin: As noted in the book (Miller and Ross 1993), the earliest work that ultimately led to what is now called the Riemann-Liouville definition appears to the paper by N.Ya. Sonin (1869) “On Differentiation with Arbitrary Index”. His starting point was Cauchy’s integral formula. Also, A.V.Letnikov wrote four papers on this topic from 1868 to 1872. His paper ‘An explanation on the main concept of the theory of differentiation of arbitrary index’ (1872) is an extension of Sonin’s paper. The contributions of Abel and Liouville, Leibniz, Euler, Laplace, Lacroix Fourier made mention of derivatives of arbitrary order, but the first use of fractional operations was by Niels Henrik Abel in 1823 [Abel 1881]. Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone (isochrone) problems. H.Laurent (1884) introduced integration along an open circuit C on Riemann surface, in contrast to the closet circuit C_0 of Sonin and Letnikov. The final form of this concept represented by K.Nishimoto (1984-1994) [32]. P.A.Nekrassov (1888), A.Krug (1890) also obtained the fundamental definition from Cauchy’s integral formula, their method differing in choice of a contour of integration. It remains a curious fact, however, these generalized operators of integration and their connection with the Cauchy integral formula have succeeded in securing for themselves, to this day, only passing references in standard works in the theory of analytic function [34]. S.Priyadharsini works on the recent stability results of fractional differential equations and the analytical method used. She discuss the stability of the linear fractional system by analyzing the eigen values, also the stability of the non-linear dynamical system, by giving conditions on the non-linear term. Further she study the stability of fractional neutral and integro differential systems [35]. Liouville (1832a) was expanded functions in series of exponentials and defined the q^{th} derivative of such a series by operating term-by-term as though q were a positive integer. Riemann (1953) proposed a different definition that involved a definite integral and was applicable to power series with non-integer exponents. Evidently, it was Grunwald (1867) and Krug who first unified the results of Liouville and Riemann. Grunwald (1867) disturbed by the restrictions of Liouville’s approach, adopted on his starting point the definition of a derivative as a limit of difference quotient and arrived as definite-integral formulas for the q^{th} derivative. Krug (1890), working through Cauchy’s integral formula for ordinary derivatives, showed that Riemann’s definite integral had to be interpreted as having a finite lower limit while Liouville's definition, in which no distinguishable lower limit appeared, corresponded to a lower limit -∞ [23]. G.W.Leibniz, letter from Hanover, Germany, May 28, 1697 to J. Wallis. In this letter Leibniz discusses Wallis infinite product for Π. Leibniz mentions differential calculus and uses of the notation $d^{\frac{1}{2}}y$ to denote a derivative of order $\frac{1}{2}$. In 1819, S.F.Lacroix, Traite du Calcul Differentiel et du Calcul Integral, 2nd ed.,Vol. 3,pp 409-410 Courcier, Paris. In this 700 page text two paper was devoted to fractional calculus. Lacroix develops a formula for fractional differentiation for the n^{th} derivative of v^m by induction. Then, he formerly replaces *n* with the fraction $\frac{1}{2}$. In 1839, S.S.Greatheed, " On General Differentiation No.I", Cambridge Math, J.I.,pp.11-12. In the same issue are two more papers: "On General Differentiation No.II", Cambridge Math, J.I.,pp.109-117; "On the Expantion of the Function of a Binomial", Cambridge Math, J.I.,pp.67-74. In the first two papers above, Greatheed uses Liouville's definition to develop formulas for fractional differentiation. In *3rd* paper he suppliments Taylor's theorem by use of fractional derivatives. In 1847, B. Riemann, "Ver such einer Auffassung der Integration and Differentiation." Grsammelte Werke, 1876,ed. Publ. Postumously, p.331-344; 1892 ed. Pp.353-366, Teubner,Leipzig. Also,in collected works (H.Weber ed.) pp.354-360, Dover NewYork 1953. Riemann saught a generalization of a Taylor's series expansion and derived the following definition for fractional integration:

$$\frac{d^{-r}}{dx^{-r}}\, u(x) = \frac{1}{\Gamma(r)} \int_c^x (x-k)^{r-1}\, u(k)dk$$

However, he saw fit to add a complimentary functioin to the above definition. Today this definition is in common use as a definition for fractional integration but with the complimentary function taken to be identically zero and the lower limit of integration *c* is usually zero.

In 1880, A. Caley, "Note on Riemann's paper."Math.Ann.16.81-82. Referring to Riemann's paper (1847) he says, "The greatest difficulty in Riemann's theory, it appears to me, is the interpretation of a complimentary function containing an infinity of arbitrary constants." The question of the existence of a complimentary function caused much confusion. Liouville and Peacock were led in to errors and Riemann became inextricably entangled in his concept of a complimentary function. In 1880, Oliver Heaviside developed independently his operational calculus, a technique by which problems with differential equations are transformed in to algebraic equations with a differential operator p. Heaviside defined also fractional powers of p, thus establishing a connection between operation calculus and fractional calculus.[6].In 1884, H. Laurent, " Surle Calcul des derivees a idices quelconques." Nouc. Ann. Math.{3}, 3, 240-252.

Laurent generalizes Cauchy's integral formula. He does work on the generalized product rule of Leibniz but leaves the result in integral form. In 1917, G.H.Hardy published the paper "On Some Properties of Integrals of Fractional Order." Messenger Math. 47, 145-150. In 1919, E.Post, "Discussion of problems #360 and #433." Amer. Math. Monthly 26, 37-39. When two different solutions are presented to problem #433, Post takes the opportunity to answer problem #360 at the same time. He explains that the two solutions are correct; however each solution is based upon a different definition. The proposer, in his solution, used Liouville's definition of integration of fractional order which is equivalent to the definite integral

$${}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-k)^{\nu-1}\, f(t)dt$$

With lower limit of integration *c* being negative infinity while Post, in his solution used Riemann's definition which is the above integral with *c* equal to zero. Although, Post makes no reference to Center (1848[a]), it is clear why Center, with $f(x)$ equal to constant, would have to different results for the arbitrary derivative. In 1924, H.T.Davis, "Fractional operations as applied to a class of Voltera Integral Equations." Amer.J.Math, 46, 95-109. The lack of detailed explanation, understandable in a journal article, is made up for by a review of the theory of fractional calculus before the theory i9s applied to the solution of certain integral equations. This paper and Davis 1927 article are in the view of the present writer, distinguished not only for their contributions to the theory and applications of fractional calculus but also as examples of how mathematics paper should be written. In 1928, G.H.Hardy and J.E.Littlewood, "Some Properties of Fractional Integrals-I." Math.Z.27, 565-

606 (1928): “Some Properties of Fractional Integrals-II.” Math.Z. 34, 403-439 (1932). In part-I, their purpose is to develop properties of the Riemann-Liouville integral and derivative of arbitrary order of functions of certain standard classes, in particular the “Lebesgue class L^p”. Part-II is an extension of the first paper to the complex field.

In 1935, A.Zygmund, Trignometric series, Vol.II, I^st ed.Z. Subwencji Funduszu Kultury rarodewej, Warsaw; II^nd ed. Cambridge University Press, Cambridge,1959,pp.132-142.

In the section entitled “Fractional Integration”, Zygmund considers a definition of fractional integration introduced by Weyl more convenient for trigonometric series. However, it may be considered a novel topic as well, since only from a little more than twenty years it has been object of specialized conferences and treatises. For the first conference the merit is ascribed to B. Ross who organized the first conference on fractional calculus and its applications at the university of New Haven in June 1974, and edited the proceedings, see [26]. For the first monograph the merit is ascribed to K.B. Oldham and J. Spanier, see [24], who, after a joint collaboration started in 1968, published a book devoted to fractional calculus in 1974. Nowadays, the list of texts and proceedings devoted solely or partly to fractional calculus and its

applications includes about a dozen of titles [11,12,13,16,17,18,21,22,24,26,28,30,31], among which the encyclopaedic treatise by Samko, Kilbas & Marichev [30] is the most prominent. Furthermore, we recall the attention to the treatises by Davis [5], Erd´elyi [7], Gel’fand & Shilov [9], Djrbashian [3,4], Caputo [2], Babenko [1], Gorenflo & Vessella [8], which contain a detailed analysis of some mathematical aspects and/or physical applications of fractional calculus, although without explicit mention in their titles. From 1975 to 1985 only few books are available on fractional calculus which are given below:

1. Keith, B. Oldham, J. Spanier, “The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order”, Dover Books on Mathematics,1974.
2. B. Ross (Editor) Fractional Calculus and its Applications: Proceedings of the Int. Conf. held at the University of New Haven, June 1974 (Lecture notes in Mathematics), 1975.
3. Ian N. Sneddon, The Use of Operators of Fractional Integration in Applied Mathematics (Applied Mathematics Series), Polish Scientific Publishers, 1979.

3 RECENT DEVELOPMENT:

Recently, many national and international mathematicians contributed in the development of fractional calculus. Many of them have solved linear and non-linear fractional partial differential equations by developing iterative methods and finite difference methods. In 2011, A.P.Bhadane and K.C.Takale were published the paper “ Basic developments of fractional calculus and its applications”, Buletin of the Marathwada Mathematical Society, Vol.12, No. 2, Dec 2011, pp.1-7. In this paper they developed the basic theory and applications of fractional calculus. They have obtain fractional integral and fractional derivative of some functions. The fractional integral and fractional derivative of these functions are simulated by mathematical software Mathematica. In 2016,Manoj Kumar and Anuj Shankar Saxena were published the paper “Recent Advancement in Fractional Calculus”. In this paper they devoted as mathematical modeling of various real life scenarios in engineering and sciences leads to differential equation. These traditional models based on integer order derivative may introduce large errors. Fractional calculus helps in reducing this error using fractional derivatives and has compabilities to provide excellent depiction of memory and heredity properties of processes. In this review paper, they present the expressive power of fractional calculus by analyzing two examples viz., mortgage problem and fractional oscillator. These examples help in justifying the advantage of fractional calculus over its integer counter part. They also present the state of the art of fractional calculus by reviewing the rapid growth of its applications in various domains. [14]. In 2016, Vasily E. Tarasov was published the paper “Local Fractional Derivatives of Differentiable Functions are Integer Order Derivative or Zero”, International Journal of applied and computational Mathematics, 2016, Vol.2, No.2, pp. 195-201. In this paper, he prov that total fractional derivatives of differentiable functions are integer order derivative or zero operator. He demonstrate that the local fractional derivatives are limits of the left-sided Caputo fractional derivatives. The Caputo derivative of fractional order α of function $f(x)$ is defined as a fractional integration of order $n-\alpha$ of the derivative $f^{(n)}(x)$ of integer order n. the requirement of the existence of integer-order derivatives allows us to conclude that the local fractional derivative cannot be considered as the best method to describe nowhere differentiable functions and fractal objects. He also prove that unviolated Leibniz rule cannot hold for derivatives of order $\propto \neq 1$ and etc. Following is list of books which are available on fractional calculus from 1985:

1. Denis Matignon, Gérard Montseny (Editors), Fractional Differential Systems: Models, Methods and Applications, European Society for Applied and Industrial Mathematics (ESAIM), Vol. 5, 1998.
2. Journal of Vibration and Control, Special Issue: Fractional Differentiation and its Applications, vol. 14, Sept. 2008.
3. Physica Scripta Fractional Differentiation and its Applications, T136, 2009.
4. Computers and Mathematics with Applications, Special issues: - Advances in Fractional Differential Equations, vol. 59, Issue 3, pp. 1047-1376, February 2010.
5. Fractional Differentiation and its Applications, vol. 59, Issue 5, March 2010.

4 APPLICATIONS OF FRACTIONAL CALCULUS:

In the first application of semi-derivative (derivative of order$\frac{1}{2}$) is done by Abel 183 [18,23].This application of Fractional Calculus is in relation with the solution of the integral equation for the tautochrone problem. That problem deals with the determination of the slope of the curve such that the time of gravity is independent of the starting point. The last decades prove that derivatives and integrals of arbitrary order are very convenient for describing properties of real materials, e.g. polymers [25]. The new fractional order models are more satisfying than former integer-order ones. Fractional derivatives are an excellent tool for describing the memory and hereditary properties of various materials and processes while in integer-order models such effects are neglected. Recently, applications of fractional calculus are found in various fields as: viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, biology, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, percolation,modeling and identification, telecommunications, chemistry, irreversibility, physics, control systems and economics and finance.

5 C ONCLUSION AND FUTURE SCOPE

The idea of fractional calculus was born more than 300 years ago, and recently serious efforts have been dedicated to its study. Still, classical calculus is much more familiar and more preferred, may be because of its applications are more apparent. There are some gaps in the classical calculus and these can be filled by fractional calculus. Therefore fractional calculus has the potential of presenting, integrating and useful applications in the future.

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