A Class Of Univalent Analytic Functions With Fixed Second And Third Coefficients

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Abstract: In this paper we defined a new class of univalent and analytic functions with fixed second and third Taylor coefficients. Coefficient condition, starlikeness and convexity, extreme points, growth and distortion properties for this class are investigated.

***IndexTerms* – Univalent function**

I. INTRODUCTION

Let S be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let T be the subclass of functions of S which are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \; n = 2, 3, \dots \qquad (1)$$

in U and C be the subclass of functions of T which are convex in U. We have $f \in C$ if and only if $zf' \in T$.
Now we introduce a subclass $T(b, c, B_n) \subseteq T$ by fixing a_2 and a_3, for $0 \leq b \leq \frac{1}{4}$, $0 \leq c \leq \frac{1}{12}$ and $B_n \geq n(n+1)$ for $n \geq 2$,

$$T(b, c, B_n) = \{f(z) \in T : f(z) = z - bz^2 - c\,z^3 - \sum_{n=4}^{\infty} a_n z^n, \; \sum_{n=3}^{\infty} B_n a_{n+1} \leq 2b - c\,B_2\}.$$

Let $C(b, c, B_n)$ be a subclass of functions of $T(b, c, B_n)$ which is convex in U.

This paper consists of two sections. In section 1, we find the coefficient conditions for starlikeness and convexity of the class $T(b, c, B_n)$. In section 2 we find extreme points, growth and distortion properties for the class $T(b, c, B_n)$.

SECTION 1

We need the following definitions from [1].

Definition1: [1] A function $f(z) \in S$ is said to be starlike of order α $(0 \leq \alpha < 1)$ in U, if it satisfies the inequality $Re\left[\frac{z\,f'(z)}{f(z)}\right] > \alpha$ for $z \in U$. The class of starlike functions of order α is denoted by $S^\star(\alpha)$.
Definition 2: [1] A function $f(z) \in S$ is said to be convex of order α $(0 \leq \alpha < 1)$ in U, if it satisfies the inequality $Re\left[1 + \frac{z\,f''(z)}{f'(z)}\right] > \alpha$ for $z \in U$. The class of convex functions of order α is denoted by $C^\star(\alpha)$.
We have $f \in C^\star(\alpha)$ if and only if $zf' \in S^\star(\alpha)$.

We start with a coefficient characterization for the functions of T to be in the class $T(b, c, B_n)$.

Theorem-1
The function $f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n z^n$, $z \in U$ is in the class $T(b, c, B_n)$ if and only if $\sum_{n=3}^{\infty} n(n+1)\, a_{n+1} \leq 2b - 6c$. The result is sharp.

Proof: If $f(z) = z - b\,z^2 - c\,z^3 - \sum_{n=4}^{\infty} a_n z^n$, $z \in U$ belongs to the class $T(b, c, B_n)$,
Then by the definition, we have $\sum_{n=3}^{\infty} B_n a_{n+1} \leq 2b - cB_2$
This gives $\sum_{n=3}^{\infty} n(n+1)\, a_{n+1} \leq 2b - cB_2$
or $\sum_{n=3}^{\infty} n(n+1)\, a_{n+1} \leq 2b - c.2.3$
this shows $\sum_{n=3}^{\infty} n(n+1)\, a_{n+1} \leq 2b - 6c$ (2)

Now, suppose that $\sum_{n=3}^{\infty} n(n+1)\, a_{n+1} \leq 2b - 6c$
Then $\sum_{n=2}^{\infty} n\, a_n \leq 1$.
Therefore $f(z) \in T$ by [3].

Since $B_n \geq n(n+1)$ for $n \geq 2$, we obtain

$$\sum_{n=3}^{\infty} B_n\, a_{n+1} \leq B_n \frac{2b-6c}{n(n+1)} \leq 2b-6c \leq 2b - cB_2$$

This shows that $f(z) \in T(b,c,B_n)$.

Sharpness of the result occurs by taking $f_n(z) = z - b\,z^2 - c\,z^3 - \frac{2b-6c}{n(n+1)} z^{n+1}, n \geq 3.$

Corollary: If $f(z) = z - b\,z^2 - c\,z^3 - \sum_{n=4}^{\infty} a_n\, z^n \in T(b,c,B_n)$ for $z \in U$, then $a_n \leq \frac{2b-6c}{n(n-1)}$ for $n \geq 4$. The result is sharp.

Proof: From (2) we have $a_{n+1} \leq \frac{2b-6c}{n(n+1)}$ for $n \geq 3$

Thus $a_n \leq \frac{2b-6c}{n(n-1)}$ for $n \geq 4$ (3)

By taking the function $f(z)$ of the form $f_n(z) = z - bz^2 - cz^3 - \frac{2b-6c}{(n-1)n} z^n,\ n \geq 4$, we see that the result (4) is sharp.

In the following result we present a sufficient condition for a function in $T(b,c,B_n)$ to be starlike in U.

Theorem-2

A function $f(z) = z - b\,z^2 - c\,z^3 - \sum_{n=4}^{\infty} a_n\, z^n$ belonging to $T(b,c,B_n)$ for $z \in U$, is starlike of order α where $0 \leq \alpha < 1$ if $\sum_{n=4}^{\infty}(n-\alpha)a_n \leq (1-\alpha) - (2-\alpha)b - (3-\alpha)c.$

The result is sharp for $f_n(z) = z - b\,z^2 - c\,z^3 - \frac{(1-\alpha)-(2-\alpha)b-(3-\alpha)c}{n-\alpha} z^n,\ n \geq 4.$

Proof: For $z \in U$, we have

$$\left|\frac{z\,f'}{f} - 1\right| = \left|\frac{z\,f' - f}{f}\right|$$
$$= \left|\frac{-bz^2 - 2cz^3 - \sum_{n=4}^{\infty}(n-1)a_n\, z^n}{1 - bz - cz^3 - \sum_{n=4}^{\infty} a_n\, z^{n-1}}\right|$$
$$= \left|\frac{-(bz + 2cz^2 - \sum_{n=4}^{\infty}(n-1)a_n\, z^n)}{1 - bz - cz^3 - \sum_{n=4}^{\infty} a_n\, z^{n-1}}\right| \leq \frac{b + 2c + \sum_{n=4}^{\infty}(n-1)a_n}{1 - b - c - \sum_{n=4}^{\infty} a_n}$$

Now, the hypothesis of the theorem gives

$$\left|\frac{z\,f'}{f} - 1\right| \leq 1 - \alpha$$

when $\sum_{n=4}^{\infty}(n-\alpha)a_n \leq (1-\alpha) - (2-\alpha)b - (3-\alpha)c.$ (4)

This final inequality is the given condition and hence the proof is complete.

Corollary: A function $f(z) = z - b\,z^2 - c\,z^3 - \sum_{n=4}^{\infty} a_n\, z^n$ belonging to $T(b,c,B_n)$ for $z \in U$, is starlike if $\sum_{n=4}^{\infty} n a_n \leq 1 - 2b - 3c.$

The result is sharp for $f_n(z) = z - b\,z^2 - c\,z^3 - \frac{1-2b-3c}{n} z^n, n \geq 4.$ (5)

In the next result we present a sufficient condition for a function in $T(b,c,B_n)$ to be convex of order α in U.

Theorem-3

A function $f(z) = z - b\,z^2 - c\,z^3 - \sum_{n=4}^{\infty} a_n\, z^n$ belonging to $T(b,c,B_n)$, for $z \in U$ is in $C(\alpha)$ for $0 \leq \alpha < 1$, if $\sum_{n=4}^{\infty} n(n-\alpha)a_n \leq (1-\alpha) - (2-\alpha)2b - (3-\alpha)3c.$

The result is sharp for $f_n(z) = z - b\,z^2 - c\,z^3 - \frac{(1-\alpha)-(2-\alpha)2b-(3-\alpha)3c}{n(n-\alpha)} z^n,\ n \geq 4.$

Proof: For $z \in U$, we have

$$\left|\frac{z\,f''(z)}{f'(z)}\right| = \left|\frac{-2bz - 6cz^2 - \sum_{n=4}^{\infty} n(n-1)a_n\, z^{n-2}}{1 - 2bz - 3cz^2 - \sum_{n=4}^{\infty} n a_n\, z^{n-1}}\right|$$
$$\leq \frac{2b + 6c + \sum_{n=4}^{\infty} n(n-1)a_n}{1 - 2b - 3c - \sum_{n=4}^{\infty} n a_n}$$

Then $\left|\frac{z\,f''(z)}{f'(z)}\right| \leq 1 - \alpha$

if $\sum_{n=4}^{\infty} n(n-\alpha)a_n \leq (1-\alpha) - (2-\alpha)2b - (3-\alpha)3c.$ (6)

This final inequality is the given condition and hence the proof is complete.

Corollary:

A function $f(z) = z - b\,z^2 - c\,z^3 - \sum_{n=4}^{\infty} a_n\, z^n \in\, T(b, c, B_n),\; z \in\, U$ is in C if $\sum_{n=4}^{\infty} n^2 a_n \;\le\; 1 - 4b - 9c.$

Sharpness occurs for $f_n(z) = z - b\,z^2 - c\,z^3 - \frac{1-4b-9c}{n^2} z^n,\; n \ge 4.$ (7)

Section 2

In the first result we show that $T(b, c, B_n)$ is a convex family.

Theorem-4

The class $T(b, c, B_n)$ is a convex subfamily of T.

Proof: Let $f, g \in\, T(b, c, B_n)$

and $f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n\, z^n$,

$g(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} b_n\, z^n$

$F(z) = \lambda f(z) + (1-\lambda) g(z)$

$= z - bz^2 - cz^3 - \sum_{n=4}^{\infty} [\lambda a_n + (1-\lambda) b_n]\, z^n$

$= z - bz^2 - cz^3 - \sum_{n=4}^{\infty} A_n\, z^n$ where $A_n = \lambda a_n + (1-\lambda) b_n$ for $0 \le \lambda \le 1$

$\sum_{n=3}^{\infty} B_n\, A_{n+1} = \sum_{n=3}^{\infty} B_n\, [\lambda a_{n+1} + (1-\lambda) b_{n+1}]$

$= \lambda \sum_{n=3}^{\infty} B_n a_{n+1} + (1-\lambda) \sum_{n=3}^{\infty} B_n\, b_{n+1}]$

$\le \lambda(2b - cB_2) + (1-\lambda)(2b - cB_2)$ (since $f, g \in T(b, c, B_n)$)

$= (2b - cB_2)$

This shows that $F(z) \in\, T(b, c, B_n)$.

Hence $T(b, c, B_n)$ is a convex subfamily of T.

In the next result we find the extreme points for the class $T(b, c, B_n)$.

Theorem-5

Let $B_n \ge \frac{(n+1)(2b-6c)}{1-2b-3c} > 0$ for $n \ge 2$, $0 < b \le \frac{1}{4}$ and $0 < c \le \frac{1}{12}$,

$f_2(z) = z - bz^2 - cz^3$ (8)

$f_n(z) = z - bz^2 - cz^3 - \frac{2b-6c}{B_n} z^{n+1}$ (9)

for $z \in\, U, n \ge 3$. Then $f \in\, T(b, c, B_n)$ if and only if $f(z)$ can be expressed as

$f(z) = \sum_{n=2}^{\infty} \lambda_n\, f_n(z),\; z \in\, U,$

where $\lambda_n \ge 0$ for $n \ge 2$ and $\sum_{n=2}^{\infty} \lambda_n = 1$

Proof: Assume that $f(z)$ can be expressed in the form (10). Then

$f(z) = \sum_{n=2}^{\infty} \lambda_n\, f_n(z)$

$= \lambda_2 f_2(z) + \lambda_3 f_3(z) + \cdots + \lambda_n f_n(z) + \cdots$

$= \lambda_2(z - bz^2 - cz^3) + \lambda_3\left(z - bz^2 - cz^3 - \frac{2b-6c}{B_3} z^4\right) + \cdots$

$+\lambda_n\left(z - bz^2 - cz^3 - \frac{2b-6c}{B_n} z^{n+1}\right) + \cdots$

$= (\lambda_2 + \lambda_3 + \cdots + \lambda_n + \cdots)(z - bz^2 - cz^3) + \left(\frac{\lambda_3}{B_3} z^4 + \cdots + \frac{\lambda_n}{B_n} z^{n+1} + \cdots\right)(2b - 6c)$

$= z - bz^2 - cz^3 - \sum_{n=3}^{\infty} \frac{(2b-6c)\lambda_n}{B_n} z^{n+1}$

$= z - \sum_{n=2}^{\infty} A_n z^n,$ where $A_n = \frac{(2b-6c)\lambda_n}{B_n}$ for $n \ge 3$ and $A_2 = b, A_3 = c.$

Here $\sum_{n=2}^{\infty} nA_n \le 1$

This shows that $f(z) \in\, T.$

and $\sum_{n=3}^{\infty} B_n\, A_{n+1} = \sum_{n=3}^{\infty} B_n \frac{2b-6c}{B_{n+1}} \lambda_{n+1}$

$= (2b - 6c) \sum_{n=3}^{\infty} \frac{B_n}{B_{n+1}} \lambda_{n+1} \le 2b - 6c$

which implies that $f(z) \in\, T(b, c, B_n)$.

Conversely, suppose that $f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n\, z^n$ belongs to the class $T(b, c, B_n)$.

Therefore,

$\sum_{n=3}^{\infty} B_n a_{n+1} \le 2b - 6c$ for $n \ge 3$

For $2b \neq 6c$, if we set

$\lambda_n = \frac{B_n a_{n+1}}{2b-6c} \geq 0$ for $n \geq 3$, we have

$\sum_{n=2}^{\infty} \lambda_n = 1$ shows $\lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n$

Then

$$f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n z^n$$
$$= \lambda_2(z - bz^2 - cz^3) + \sum_{n=3}^{\infty} \lambda_n \left(z - bz^2 - cz^3 - \frac{2b-6c}{B_n} z^{n+1}\right)$$
$$= \sum_{n=2}^{\infty} \lambda_n f_n(z)$$

Any of the functions (8), (9) cannot be expressed as a proper convex linear combination of distinct functions in $T(b,c,B_n)$. Thus extreme points of $T(b,c,B_n)$ are given by (8) and (9).

Now we find the extreme points for the class $C(b,c,B_n)$.

Corollary

Let $B_n \geq \frac{(n+1)^2(2b-6c)}{1-4b-9c} > 0$, $0 < b \leq \frac{1}{4}$ and $0 < c \leq \frac{1}{12}$

$$f_2(z) = z - bz^2 - cz^3 \quad (10)$$

$$f_n(z) = z - bz^2 - cz^3 - \frac{2b-6c}{B_n} z^{n+1} \quad (11)$$

for $z \in U$, $n \geq 3$. Then $f \in C(b,c,B_n)$ for $z \in U$ if and only if $f(z)$ can be expressed as

$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z)$

where $\lambda_n \geq 0$ for $n \geq 2$ and $\sum_{n=2}^{\infty} \lambda_n = 1$.

Extreme points of $C(b,c,B_n)$ are given by (10) and (11).

Now we find growth and distortion bounds for the class $T(b,c,B_n)$.

Theorem-6

Let $f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n z^n \in T(b,c,B_n)$ for $z \in U$, where $\{B_n\}$ is a non-decreasing sequence with $B_n > 0$ for $n \geq 2$. Then for $|z| = r$ and $z \in U$,

$$max\left\{0,\ r - br^2 - cr^3 - \frac{2b-6c}{B_3} r^4\right\} \leq |f(z)| \leq r + br^2 + cr^3 + \frac{2b-6c}{B_3} r^4.$$

The lower inequality is sharp for

$f(z) = z - bz^2 - cz^3 - \frac{2b-6c}{B_3} z^4$ when $B_3 \geq \frac{4(2b-6c)}{1-2b-3c}$ and $0 < b \leq \frac{1}{4}$, $0 < c \leq \frac{1}{12}$.

Proof:

Since $f(z) \in T(b,c,B_n)$ and sequence $\{B_n\}$ is non-decreasing,

then $\sum_{n=3}^{\infty} B_n a_{n+1} \leq 2b - 6c$

this shows $\sum_{n=3}^{\infty} a_{n+1} \leq \frac{2b-6c}{B_n} \leq \frac{2b-6c}{B_3}$

$$|f(z)| \geq max\{0,\ |z| - b|z|^2 - c|z|^3 - \sum_{n=4}^{\infty} a_n |z|^n\}$$
$$\geq max\{0,\ |z| - b|z|^2 - c|z|^3 - |z|^4 \sum_{n=4}^{\infty} a_n\}$$
$$\geq max\left\{0,\ r - br^2 - cr^3 - \frac{2b-6c}{B_3} r^4\right\}$$

Also,

$$|f(z)| \leq r + br^2 + cr^3 + \sum_{n=4}^{\infty} a_n r^n$$
$$\leq r + br^2 + cr^3 + r^4 \sum_{n=4}^{\infty} a_n$$
$$\leq r + br^2 + cr^3 + \frac{2b-6c}{B_3} r^4$$

Thus we have

$$max\left\{0,\ r - br^2 - cr^3 - \frac{2b-6c}{B_3} r^4\right\} \leq |f(z)| \leq r + br^2 + cr^3 + \frac{2b-6c}{B_3} r^4.$$

Hence, the proof is complete.

Corollary:

Let $f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n z^n \in C(b,c,B_n)$ for $z \in U$, where $\{B_n\}$ is a non-decreasing sequence with $B_n > 0$ for $n \geq 2$. Then for $|z| = r$ and $z \in U$,

$$max\left\{0,\ r - br^2 - cr^3 - \frac{2b-6c}{B_3} r^4\right\} \leq |f(z)| \leq r + br^2 + cr^3 + \frac{2b-6c}{B_3} r^4 .$$

The lower inequality is sharp for

$f(z) = z - bz^2 - cz^3 - \frac{2b-6c}{B_3} z^4$ when $B_3 \geq \frac{16(2b-6c)}{1-4b-9c}$ and $0 < b \leq \frac{1}{4}$, $0 < c \leq \frac{1}{12}$.

Theorem-7

Let $f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n z^n \in T(b,c,B_n)$ for $z \in U$, where $\{B_n\}$ is an increasing sequence with $B_n > 0$ for $n \geq 2$. Then for $|z| = r$ and $z \in U$,

$$max\left\{0, 1 - 2br - 3cr^2 - \left(\frac{2b-6c}{B_3}\right) r^3\right\} \leq |f'(z)| \leq 1 + 2br + 3cr^2 + \left(\frac{2b-6c}{B_3}\right) r^3.$$

The lower inequality is sharp for

$f(z) = z - bz^2 - cz^3 - \frac{2b-6c}{4B_3} z^4$, when $B_3 \geq \frac{4(2b-6c)}{1-2b-3c}$ and $0 < b \leq \frac{1}{4}$, $0 < c \leq \frac{1}{12}$.

Proof:

By assumption, $f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n z^n \in T(b,c,B_n)$

Then by Theorem-1, we have

$$\sum_{n=3}^{\infty} n(n+1)\, a_n \leq 2b - 6c$$

this shows $\sum_{n=3}^{\infty} B_n a_{n+1} \leq 2b - 6c$

which implies $\sum_{n=3}^{\infty} a_{n+1} \leq \frac{2b-6c}{B_n} \leq \frac{2b-6c}{B_3}$

$$|f'(z)| \geq max\{0,\ 1 - 2b|z| - 3c|z|^2 - r^3 \sum_{n=4}^{\infty} n\, a_n\}$$
$$\geq max\left\{0,\ 1 - 2br - 3cr^2 - \left(\frac{2b-6c}{B_3}\right) r^3\right\}$$

Also

$$|f'(z)| \leq 1 + 2br + 3cr^2 + r^3 \sum_{n=4}^{\infty} n\, a_n$$
$$\leq 1 + 2br + 3cr^2 + \left(\frac{2b-6c}{B_3}\right) r^3$$

Therefore,

$$max\left\{0, 1 - 2br - 3cr^2 - \left(\frac{2b-6c}{B_3}\right) r^3\right\} \leq |f'(z)| \leq 1 + 2br + 3cr^2 + \left(\frac{2b-6c}{B_3}\right) r^3.$$

Hence, the proof is complete.

Corollary:

Let $f(z) = z - bz^2 - cz^3 - \sum_{n=4}^{\infty} a_n z^n \in C(b,c,B_n)$ for $z \in U$, where $\{B_n\}$ is an increasing sequence with $B_n > 0$ for $n \geq 2$. Then

$$max\left\{0, 1 - 2br - 3cr^2 - \left(\frac{2b-6c}{B_3}\right) r^3\right\} \leq |f'(z)| \leq 1 + 2br + 3cr^2 + \left(\frac{2b-6c}{B_3}\right) r^3.$$

The lower inequality is sharp for

$f(z) = z - bz^2 - cz^3 - \frac{2b-6c}{4B_3} z^4$, when $B_3 \geq \frac{16(2b-6c)}{1-4b-9c}$ and $0 < b \leq \frac{1}{4}$, $0 < c \leq \frac{1}{12}$.

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