Upper And Lower Bounds On The Chromatic Number Of S (n, m) Graphs.

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Abstract

In this paper we discuss about the lower and upper bounds on Chromatic number of $S(n,m)$ graphs. We have also discussed about the Chromatic number of $S(n,m)$ for $n \geq 2m+2$, odd $m \geq 3$, $S(n,2)$ and $S(n,4)$ graphs.

Keywords : Lower and Upper bounds, Chromatic number of $S(n,m)$ graphs.

1. Introduction

In this paper we consider the graph $S(n,m)$ which is a quartic graph and also both Eulerian and Hamiltonian.

The graph $S(n,m)$\cite{1} consists of $n$ vertices denoted as $v_1, v_2, \ldots, v_n$. The edges are defined as follows:

i) $v_i$ is adjacent to $v_{i+1}$ and $v_n$ is adjacent to $v_1$.

ii) $v_i$ is adjacent to $v_{i+m}$ if $i+m \leq n$.

iii) $v_i$ is adjacent to $v_{i+m-n}$ if $i+m > n$.

Definition 1.1. A $k$-vertex coloring or $k$-coloring for short, of a graph $G$ is an assignment of one of $k$ available colors to each vertex 'x' of $G$ such that adjacent vertices receive different colors. The smallest $k$ for which a graph $G$ admits a $k$-coloring is called the Chromatic Number of $G$ and is denoted by $\chi(G)$.

Definition 1.2. The matrix $Q = A + D$, where $A$ is the Adjacent matrix of graph $G$ and $D$ is the diagonal matrix whose main entries are the degrees in $G$, is called the Signless Laplacian of $G$.

2. BOUNDS ON CHROMATIC NUMBER OF $S(n,m)$ GRAPHS ($n \geq 2m+2$)

Theorem 2.1: The Chromatic number $\chi[S(n,m)]$, $n \geq 2m+2$ satisfies $1 + [\frac{4}{4-\delta_n}] \leq \chi[S(n,m)] \leq 5$, where $\delta_n$ is the eigenvalue of Signless Laplacian of $S(n,m)$.

Proof. In 2011 Lima, Oliveira, Abreu and Nikiforov \cite{2, 3} proved that $\chi(G) \geq 1 + [\frac{2q}{2q-p\delta_p}]$, where $G$ is a graph with $q$ edges and $p$ vertices. $\delta_p$ is the eigenvalue of Signless Laplacian of $G$ which satisfies $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_p \geq 0$.

In $S(n, m)$ graphs, the number of edges is twice the number of vertices. i.e., Number of edges = $2n$. This is illustrated by Fig.1.
By Greedy Coloring Theorem, \( \chi( G) \leq d+1 \), where \( d \) is the largest degree of the vertex. In \( S(n,m) \) graphs the degree of each vertex is 4 \( \text{FIG.1} \) and so \( \chi [S(n,m)] \leq 5 \). In particular, since, \( \delta_n \geq 0, \frac{4}{4-\delta_n} \geq 1 \) and so \( 1 + \left[ \frac{4}{4-\delta_n} \right] \geq 2 \). So, \( 2 \leq \chi [S(n,m)] \leq 5 \).

3. CHROMATIC NUMBER OF S\((n,m)\) GRAPHS

**Theorem 3.1.** The chromatic number \( \chi [S(n,m)] \) is 
(i) 2 for even \( n \geq 2m+2 \) and odd \( m \geq 3 \) 
(ii) 4 for odd \( n \geq 2m+2 \) and odd \( m \geq 3 \).

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of the graph \( S(n,m) \) and its edges be denoted by \((v_i,v_{i+1}), (v_i,v_{i+m}), (v_i,v_{i+n-m})\) for \( i = 1,2,3,\ldots \) and \((v_n,v_1)\). Let the coloring set of \( S(n,m) \) be \( \{1,2,3,\ldots\} \). We define the function \( f \) from the vertex set of \( S(n,m) \) to the coloring set \( \{1,2,3,\ldots\} \) as follows:

**Case (i) :** Even \( n \geq 2m+2 \) and odd \( m \geq 3 \).

\[
\begin{align*}
\text{Using the above pattern the graph } S(n,m) \text{ for even } n &\geq 2m+2 \text{ and odd } m \geq 3 \text{ admits vertex coloring. The chromatic number } \chi [S(n,m)] = 2. \\
\text{Case (ii) : Odd } n &\geq 2m+2 \text{ and odd } m \geq 3. \\
\end{align*}
\]

Using the above pattern the graph \( S(n,m) \) for odd \( n \geq 2m+2 \) and odd \( m \geq 3 \), admits vertex coloring. The chromatic number \( \chi [S(n,m)] = 4. \)

**Theorem 3.2.** The chromatic number of \( S(n,2) \) for \( n \geq 6 \) is

(i) 3 for \( n \equiv 0 \pmod{6} \) and \( n \equiv 3 \pmod{6} \)
(ii) 4 for \( n \equiv 1 \pmod{6} \) and \( n \equiv 4 \pmod{6} \)

(iii) 5 for \( n \equiv 2 \pmod{6} \) and \( n \equiv 5 \pmod{6} \).

Proof. Let \( v_1, v_2, \ldots, v_n \) be the vertices of the graph \( S(n,2) \) and its edges be denoted by \((v_i,v_{i+1}), (v_i,v_{i+2}), (v_i,v_{i+n-2})\) for \( i = 1, 2, 3, \ldots \) and \((v_n,v_1)\). Let \( f \) be a function that maps vertex set of \( S(n,2) \) to the coloring set \{1,2,3\ldots\}.

Case (i): \( n \equiv 0 \pmod{6} \) and \( n \equiv 3 \pmod{6} \)

\[
f(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1 \pmod{3}, 1 \leq i \leq n \\
2, & \text{for all } i \equiv 2 \pmod{3}, 1 \leq i \leq n \\
3, & \text{for all } i \equiv 0 \pmod{3}, 1 \leq i \leq n 
\end{cases}
\]

By using above pattern \( S(n,2) \) admits vertex coloring. The chromatic number \( \chi[S(n,2)] = 3 \).

Case (ii): \( n \equiv 1 \pmod{6} \) and \( n \equiv 4 \pmod{6} \)

\[
f(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1 \pmod{3}, 1 \leq i \leq n - 1 \\
2, & \text{for all } i \equiv 2 \pmod{3}, 1 \leq i \leq n - 1 \\
3, & \text{for all } i \equiv 0 \pmod{3}, 1 \leq i \leq n - 1 
\end{cases}
\]

\( f(v_n) = 4 \). Using the above pattern \( S(n,2) \) admits vertex coloring. The chromatic number \( \chi[S(n,2)] = 4 \).

Case (iii): \( n \equiv 2 \pmod{6} \) and \( n \equiv 5 \pmod{6} \)

\[
f(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1 \pmod{3}, 1 \leq i \leq n - 2 \\
2, & \text{for all } i \equiv 2 \pmod{3}, 1 \leq i \leq n - 2 \\
3, & \text{for all } i \equiv 0 \pmod{3}, 1 \leq i \leq n - 2 
\end{cases}
\]

\( f(v_{n-1}) = 4 \) and \( f(v_n) = 5 \). Using the above pattern \( S(n,2) \) admits vertex coloring. The chromatic number \( \chi[S(n,2)] = 5 \).

**Theorem 3.3.** The chromatic number of \( S(n,4) \), \( n \geq 10 \) is 3 for \( n \equiv 0 \pmod{3} \), \( n \equiv 2 \pmod{3} \) and \( n \equiv 1 \pmod{3} \).

Proof. Let \( v_1, v_2, \ldots, v_n \) be the vertices of the graph \( S(n,4) \) and its edges be denoted by \((v_i,v_{i+1}),(v_i,v_{i+4}),(v_i,v_{i+n-4})\) for \( i = 1, 2, 3, \ldots \) and \((v_n,v_1)\). Let \( f \) be a function that maps vertex set of \( S(n,4) \) to the coloring set \{1,2,3\ldots\}.

Case (i): \( n \equiv 0 \pmod{3} \) and \( n \equiv 2 \pmod{3} \)

\[
f(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1 \pmod{3}, 1 \leq i \leq n \\
2, & \text{for all } i \equiv 2 \pmod{3}, 1 \leq i \leq n \\
3, & \text{for all } i \equiv 0 \pmod{3}, 1 \leq i \leq n 
\end{cases}
\]

Using the above pattern \( S(n,4) \) admits vertex coloring. The chromatic number \( \chi[S(n,4)] = 3 \).

Case (ii): \( n \equiv 1 \pmod{3} \)

\[
f(v_i) = \begin{cases} 
1, & \text{for all } i \equiv 1 \pmod{3}, 1 \leq i \leq n - 4 \\
2, & \text{for all } i \equiv 2 \pmod{3}, 1 \leq i \leq n - 4 \\
3, & \text{for all } i \equiv 0 \pmod{3}, 1 \leq i \leq n - 4 
\end{cases}
\]
f(v_{n-3}) = 2, \ f(v_{n-2}) = 3, \ f(v_{n-1})= 1 \text{ and } f(v_n) = 2. \text{ Using the above pattern } S(n,4) \text{ admits vertex coloring. The chromatic number } \chi [S(n,4)] = 3.

4. **Conclusion.** We have found the lower and upper bounds on chromatic number of \( S(n,m), n \geq 2m+2 \). In general \( \chi [S(n,m)], n \geq 2m+2 \) satisfies \( 2 \leq \chi [S(n,m)] \leq 5 \). The chromatic number of \( S(n,m), n \geq 2m+2, \) when \( m = 2,3,4 \) are also discussed.

5. **References**

