Automorphism Group of Strongly Regular Integral Circulant Graphs

S. Raja

Department of Mathematics, SCAD College of Engineering and Technology,

Cheranmahadevi, Tamilnadu, India.

Email: nellairajaa@yahoo.com

Abstract

Let $S \subseteq \mathbb{Z}_n$, the finite cyclic group of order $n > 1$. Assume that $0 \notin S$ and $-S = \{s: s \in S\}$. The Circulant graph G= Cir (n, S) is the undirected graph having the vertex set V(G)= $\mathbb{Z}_n$ and edge set E(G)= $\{ab: a, b \in \mathbb{Z}_n$, a-b∈ S}. In this paper, we deal with the automorphism group of strongly regular integral circulant graphs. We determine the size and the structure of the automorphism group of the certain integral circulant graphs.

2000 Mathematics Subject Classification: *05C50, 05C25.*
Keywords: *Cayley graphs, Integral circulant graph, automorphism.*

Introduction

Let G be an undirected finite simple graph with *n* vertices and adjacency matrix A(G). Since A(G) is a real symmetric matrix, its eigenvalues are real numbers. Without loss of generality one can assume that the eigenvalue set of A (G) is $\{\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n\}$, such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. A graph is integral if all the eigenvalues of its adjacency matrix are integers. The notion of integral graphs dates back to Harary and Schwenk [9]. Since then, many integral graphs have been discovered and for a survey see [1]. The problem of characterizing integral graphs seems to be very difficult and so it is wise to restrict ourselves to certain families of graphs. In this paper, we determine the order of the automorphism group of certain strongly regular integral circulant graphs.

The concept of Cayley graphs was introduced by Arthur Cayley, aimed at explaining the concept of abstract groups that are described by a set of generators. The Cayley graph $Cay(\Gamma, \Omega)$ of a group Γ with identity 1 and a set $\Omega \subseteq \Gamma$ is defined to have vertex set Γ and edge set $\{a, b \in \Gamma, ab^{-1} \in \Omega\}$. The set Ω is usually assumed to satisfy $1 \notin \Omega$ and $\Omega = \Omega^{-1} = \{a; a^{-1} \in \Omega\}$ which implies that $Cay(\Gamma, \Omega)$ is loop-free and undirected. For general properties of Cayley graphs we refer to Godsil and Royle [6].

Circulant graphs are Cayley graphs on finite cyclic groups, and found applications in telecommunication networks and distributed computation [3]. Recall that for a positive integer n and a subset *S* ⊆ *{0,1,2,…,n-1}*, the circulant graph *G (n, S)* is the graph with *n* vertices, labeled with integers modulo n, such that each vertex i is adjacent to |S| other vertices {i+s(mod n)| $s \subseteq$ S}. Wasin So [15] studied integral circulant graphs. Actually he proved that there are exactly $2^{\tau(n)-1}$ non-isomorphic integral circulant graphs on n vertices, where $\tau(n)$ is the number of divisors of n.

In this paper, we characterize the automorphism group Aut(Cir(n, S)) of certain strongly regular integral circulant graphs, and make a step towards characterizing the automorphism group of an arbitrary integral circulant graph. Many authors studied the isomorphisms of circulant and Cayley graphs [11,14], automorphism groups of Cayley digraphs [4], integral Cayley graphs over abelian groups [12], etc. For the survey on the automorphism groups of circulant graphs one can see

[13]. Following Dobson and Morris [5], dealt two cases accordingly to $n = p^k$ being a prime power and $n = p_1p_2 \dots p_k$, being a square-free number.

This paper is organized as follows. In section 2 we present some preliminary definitions and results on integral circulant graphs, while in section 3, we obtain the automorphism group of certain circulant graphs.

2 Preliminaries

First let us recall some definitions and basic results.

For a positive integer *n*, and subset $S \subseteq \{0,1,2,\dots,n-1\} \subseteq \mathbb{Z}_n$ with $S = -S$, the circulant graph $Cir(n,S)$ is the graph with vertex set $\mathbb{Z}_n = \{0,1,2,\dots,n-1\}$ and two vertices $i, j \in \mathbb{Z}_n$ are adjacent if and only if $i - j \in S$. Let $G_n(d) = \{k : \gcd(k,n) = d, 1 \leq k \leq n-1\}$ be the set of all positive integers less than n having the same greatest common divisor d with n.

Theorem 2.1. [15] A circulant graph $G = Cir(n,S)$ is integral if and only if $S = \bigcup_{d \in D} G_n(d)$ for some set of divisors D and $0 \notin S$ *and* $-S = \{s : s \in S\} = S$.

Definition 2.2. Two graphs $G = (V,E)$ and $H = (V',E')$ are said to be isomorphic if there is a bijective mapping ϕ from the vertex set V to the vertex set V' such that (u,v) ∈ E(G) if and only if $(\phi(u),\phi(v)) \in E'(H)$, The mapping ϕ is called an isomorphism. We denote the fact that G and H are isomorphic by $G \cong H$.

That is, an isomorphism between two graphs is a bijection on the vertices that preserves edges and non edges. This definition has the following special case:

Definition 2.3. An automorphism of a graph is an isomorphism from the graph to itself.

Definition 2.4. The set of all automorphisms of a graph G forms a group, denoted Aut(G), the automorphism group of G.

Definition 2.5. A k-regular graph G with n vertices is said to be strongly regular with parameters (n,k,a,c) if the following conditions are obeyed [2].

- G is neither complete, nor empty;
- any two adjacency vertices of G have a common neighbors;
- any two nonadjacent vertices of G have c common neighbors.

We assume throughout that the considered strongly regular graph G is connected.

Lemma 2.6. Let G be a connected regular graph with exactly three distinct eigenvalues. Then G is strongly regular.

3 Main Results:

Let n > 1 be a composite integer and D = $\{d_1, d_2, \dots d_k\}$ be the set of all positive, proper integer divisors of n. Let $M_n(d) = \{d, 2d, 3d, \dots, n-d\}$ be set of multiple of d less than n. In this section, we consider circulant graphs $G = Cir(n, S)$, where $S = \mathbb{Z}_n^* - M_n(d)$ for a fixed $1 < d \in D$.

Lemma 3.1. [16] Let d be a proper divisor of a positive composite integer n and $M_n(d) = \{d, 2d, \dots, n-d\}$. If $S = \mathbb{Z}_n^* - M_n(d)$ and $G = Cir(n, S)$, then G is an integral circulant graph.

Proof. Clearly $\mathbb{Z}_n^* = \bigcup_{d\in D} G_n(d)$ for all d, $1 \le d \le n-1$. Let G = Cir(n, S),where S = $\mathbb{Z}_n^* \setminus M_n(d)$. It means that S = $\bigcup_{d\in D'} G_n(d)$, where D′ = D \{d, 2d, . . . , n − d}. Then by Theorem 2.1, G = Cir(n, S) is integral.

Theorem 3.2. [16] Let d be a proper divisor of a positive composite integer n, S = $\mathbb{Z}_n^* \setminus$ {d, 2d . . . n − d} and G = Cir(n, S). Then G is a strongly regular graph with parameters $\left(n, n-\frac{n}{d}, n-\frac{2n}{d}, n-\frac{n}{d},\right)$.

Theorem 3.3. [16] Let n be a positive composite integer and G = Cir(n, S) a (connected) circulant graph with three distinct integer eigenvalues. Then S = $\mathbb{Z}_n^* \setminus M_n(d)$ for some proper divisor **d** of **n**.

The following theorem is deal about the cardinality of the Automorphism group of strongly regular integral circulant graphs.

Theorem 3.4. Let 1 < d be a proper divisor of a positive composite integer n and $M_n(d)$ = {d, 2d, . . . , n − d}. If S = $\mathbb{Z}_n^* \setminus M_n(d)$ and G = Cir(n, S), then $|\mathrm{Aut}(\mathrm{Cir}(n,S))| = d!\left(\left(\frac{n}{d}\right)!\right)^d$.

Proof. Let $C_0, C_1, C_2, \ldots C_{d-1}$ be the classes modulo d. $C_i = \left\{j : 0 \le j < n, j \equiv i (\mathrm{mod}\, d), 0 < d \le \frac{n}{2}, 0 \le i < d\right\}$. Two vertices a and b from the Circulant graph G are adjacent if and only if $a - b \equiv k\ (\mathrm{mod}\, d)$ where $0 < k < d-1$ or equivalent to $a - b \not\equiv 0\ (\mathrm{mod}\, d)$. That is all the vertices from some class C_i are adjacent to the vertices from $V(G)/C_i$.

Let $f \in Aut(G)$ be an automorphism of G. Let a and b be two vertices from the class C_i and $f(a) \in C_j$, $0 \le i, j \le d-1$. It follows $a - b \equiv 0 (mod\ d)$, which is implies that a and b are non-adjacent and consequently $f(a)$ and $f(b)$ are not adjacent. From the above consideration $f(a) - f(b) \equiv 0\ (mod\ d)$ we conclude that $f(b)$ belongs to the same class modulo b as $f(a)$, i.e., $f(b) \in C_i$. Since we choose an arbitrary index i we get that the classes are permuted under the automorphism f. Assume that the class C_i is mapped to the class C_j. Since the vertices from the class C_i form an independent set and the restriction of the automorphism f on the vertices of C_i is a bijection from C_i to C_j, we have $C_i! = \left(\frac{n}{d}\right)!$ permutation of the vertices of the class C_i. Finally, taking into account that classes and vertices permute independently, by the product rule we get that the number of Automorphism of $G = Cir\,(n, S)$ equals $d!\left(\left(\frac{n}{d}\right)!\right)^d$.

Concluding Remark

Milan Basic and Aleksandar Ilic [10] dealt with automorphism group of Unitary Cayley graph $X_n(1)$ and completely characterized automorphism groups $Aut(X_n(1,p))$ for n being a square-free number and p a prime dividing n, and $Aut(X_n(1,p^k))$ for n being a prime power. In this paper, we determine the automorphism group of $G = Cir\,(n, S)$, where $S = \mathbb{Z}_n^* - M_n(d)$ for fixed $d > 1$. Moreover, our approach can be used for establishing some upper bounds on the size of the automorphism group of strongly regular integral circulant graphs. The idea of partitioning vertices into classes modulo d and we believe that it can be extended for the full characterization of integral circulant graphs.

References

1. K. Balinska, D. Cvetkovi'c, Z. Rodosavljevic, S. Simi'c, D. Stevanovic, A Survey on integral graphs, Publ. Elektrotehn. Fak. Ser. Mat. **13** (2003) 42--65.

2. R. B. Bapat, Graphs and Matrices, Springer & Hindustan Book Agency, London, 2011.

3. J. C. Bermond, F. Comellas, D. F. Hsu, Distributed loop computer networks: a survey, J. Parallel Distributed Comput. **24** (1995) 2--10.

4. E. Dobson, I. Kovшcs, Automorphism groups of Cayley digraphs of $\mathbb{Z}_p{}^3$, *Electron. J. Combin*. **16** (2009) #R149.

5. E. Dobson and J. Morris, On automorphism groups of circulant digraphs of square-free order, *Discrete Math*. **299** (2005) 79-8.

6. C. D. Godsil, G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.

7. I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann(Eds.), *Algebraic Combinatorics and Applications, Springer, Berlin*, 2001, pp. 196--211.

8. I. Gutman, X. Li, J. Zhang, Graph energy, in: M. Dehmer, F. Emmert--Streib (Eds.), Analysis of Complex Networks. From *Biology to Linguistics, Wiley--VCH, Weinheim*, 2009, pp. 145--174.

9. F. Harary, A. J. Schwenk, Which graphs have integral spectra?, in: R. Bary, F. Harary (Eds.), *Graphs and Combinatorics*, *Springer, Berlin*, 1974, pp. 45--51.

10. Milan Bavsic and A. Ilic, On the automorphism group of integral circulant graphs, *The Electronic journal of combinatorics*, **18** (2011), #P68.

11. C. H. Li, On isomorphisms of connected Cayley graphs, *Discrete Math*. b(1998) 109122.

12. W. Klotz, T. Sander, Integral Cayley graphs over abelian groups, *Electron. J. Combin.* **17** (2010) \#R81.

13. J. Morris, Automorphism groups of circulant graphs a survey, in A. Bondy, J. Fonlupt, J. L. Fouquet, J. C. Fournier, and J. L. Ramirez Alfonsin (Eds.), *Graph Theory in Paris (Trends in Mathematics), Birkhguser*, 2007.

14. M. E. Muzychuk, A solution of the isomorphism problem for circulant graphs, *Proc. Lond. Math. Soc. Z (3)* **88** (2004) 1-41.

15. W. So, Integral circulant graphs, *Discrete Math*., 306 (2006), 153-158.

16. T. Tamizh Chelvam, S. Raja, and and Ivan Gutman, Strongly Regular Integral Circulant Graphs and Their Energies, *Bulletin of International Mathematical Virtual Institute*, Vol. **2** (2012), 9--16.