# DIFFERENTIAL TRANSFORM METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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## Abstract

In this paper a new method for solving Goursat problem is introduced using Reduced differential transform method(RDTM). The approximate analytical solution of the problem is calculated in the form of series with easily computable components. The comparison of the methodology presented in this paper with some other well known techniques demonstrates the effectiveness and power of the newly proposed methodology.

# Keywords

Differential transform method, reduced differential transform method, Goursat problem, Partial differential equations, Adomian decomposition method(ADM), Variational iteration method(VIM).

#### **1.** Introduction

Up to now, more and more nonlinear equations were presented, which described the motion of the isolated waves, localized in a small part of space, in many fields such as hydro dynamic, plasma physics, nonlinear optic, and others. The investigation of these exact solutions of these nonlinear equations is interesting and important. In the past several decades, many authors mainly had paid attention to study solutions of nonlinear equations of various methods, such as Backlund transformation (Ablowitz and Clarkson, 1991;Coely ,2001), Darboux transformation (Wadati et al., 1975), inverse scattering method (Gardner et al., 1967), Hirota's bilinear method(Hirota, 1971), the tanh method(Malfeit, 1992) the sine cosine method (Yan, 1996; Yan and Zhang, 2000), the homogeneous balance method (Wang,1996;Yan and Zhang,2001),and the Riccati expansion method with constant coefficients(Yan,2001). Recently, an extended tanh-function method and symbolic computation are suggested in Fan (2001) for solving the newly coupled modified KdV equations to obtain four kinds of solitan solutions. This method has some merits in contrast with the tanh-function method. It not only uses a simpler algorithm to produce an algebraic system, but also can pick up singular solitan solutions with no extra effort(Fan and Zhang, 1998; Hirota and Satsuma, 1981; Malfliet, 1992; Satsuma and Hirota, 1982; Wu et al., 1999). The numerical solution of Burger's equation is of great importance due to the equation's application in the approximate theory of flow through a shock wave travelling in a viscous fluid(Cole,1951) and in the Burger's model of turbulence(Burgers, 1948). It is solved analytically for arbitrary initial conditions(Hopf, 1950). Finite element methods have been applied to fluid problems, Galerkin and petrov - Galerkin finite element methods involving a time-dependent grid(Caldwell et al., 1981; Herbst et al., 1982). Numerical solution using cubic spline global functions were developed in (Rubin and Graves, 1975) to obtain two systems or diagonally dominant equations which are solved to determine the evolution of the system. A collocation solution with cubic spline

interpolation functions used to produce three coupled sets of equations for the dependent variable and its two first derivatives (Caldwell and Hinton, 1987). Ali et al(1992) applied the finite element methods to the solution of Burger's equation. The finite element approach is applied with collocation method over a constant grid of cubic spline element. Cubic spline had a resulting matrix system which is tri-diagonal and so solved by the Thomas algorithm. Soliman (2000) used the similarity reductions for the partial differential equations to develop a scheme for solving Burger's equation. This scheme is based on similarity reductions of Burger's equations on small sub-domain. The resulting similarity equation is integrated analytically. The analytical solution is then used to approximate the flux vector in Burger's equation. The coupled system is derived by Esipov (1992). It is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity(Nee and Duan, 1998). In this paper ,we consider the standard form of the Goursat problem as provided below

 $u_{xt}=f(x,t,u,u_x,u_t), 0 \le x \le a, 0 \le t \le b$ 

u(x,0)=g(x), u(0,t)=h(t),g(0)=h(0)=u(0,0)

This equation has been examined by several numerical methods such as Runge-kutta method, finite difference method, finite elements method and Adomian Decomposition method(ADM).

We will prove the applicability and effectiveness of RDTM on solving linear and non-linear Goursat problems. The main advantage of RDTM is that it can be applied directly to the problems without requiring linearization, discretization or perturbation.

# 2.Methodology

To Illustrate the basic ideas of the DTM, we considered u(x, t) is analytic and differentiated continuously in the domain of interest, then let

(1)

$$\mathbf{U}_{k}(\mathbf{x}) = \frac{1}{k!} \left[ \frac{\partial^{k} u(x,t)}{\partial t^{k}} \right]_{t=t0}$$

Where the spectrum  $U_k(x)$  is the transformed function, which is called T-function in brief. The differential inverse transform of  $U_k(x)$  is defined as follows.

$$U(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^k$$
 (2)

Combining (1) and (2), it can be obtained that

$$U(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{\partial^k u(x,t)}{\partial t^k} \right]_{t=t0} (t-t_0)^k$$
(3)

When  $(t_0)$  is taken as  $(t_0=0)$  then equation (3) is expressed as

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{\partial^k u(x,t)}{\partial t^k} \right]_{\mathbf{t}=0} \mathbf{t}^k \tag{4}$$

and equation (2) is shown as

$$\mathbf{u}(\mathbf{x},t) = \sum_{k=0}^{\infty} U_k(\mathbf{x}) t^k \tag{5}$$

In real application, the function u(x,t) by a finite series of equation (5) can be written as,

 $\mathbf{u}(\mathbf{x},\mathbf{t}) = \sum_{k=0}^{n} U_k(\mathbf{x}) t^k$ 

usually, the values of n is decided by convergence of the series coefficients. The following theorems that can be deduced from equation (3) and equation (4) are given as,

Theorem 1: If the original function is  $u(x,t)=w(x,t)\pm v(x,t)$ , then the transformed function is  $U_k(x)=W_k(x)\pm V_k(x)$ .

Theorem 2: If the original function is  $u(x,t)=\alpha v(x,t)$ , then the transformed function is  $U_k(x)=\alpha V_k(x)$ 

(6)

Theorem 3: If the original function is  $u(x,t) = \frac{\partial^m w(x,t)}{\partial t^m}$ , then the transformed function is  $U_k(x) = \frac{(k+m)!}{k!} W_k(x)$ 

Theorem 4: If the original function is  $u(x,t) = \frac{\partial w(x,t)}{\partial x}$ , then the transformed function is  $U_k(x) = \frac{\partial}{\partial x} W_k(x)$ 

Theorem 5: If the original function is  $u(x,y,z) = \frac{\partial w(x,y,t)}{\partial y}$ , then the transformed function is  $U_k(x,y) = \frac{\partial}{\partial x} W_k(x,y)$ 

Theorem 6: If the original function is  $u(x,y,z,t) = \frac{\partial w(x,y,z,t)}{\partial z}$ , then the transformed function is  $U_k(x,y,z) = \frac{\partial}{\partial z} W_k(x,y,z)$ 

Theorem 7: If the original function is  $u(x,t)=x^mt^n$ , then the transformed function is  $U_k(x)=x^m\delta(k-n)$ 

Theorem 8: If the original function is  $u(x,t)=x^{m}t^{n}w(x,t)$ , then the transformed function is  $U_{k}(x)=x^{m}W_{k-n}(x)$ 

Theorem 9: If the original function is u(x,t)=w(x,t)v(x,t), then the transformed function is  $U_k(x)=\sum_{r=0}^k W_r(x)V_{k-r}(x)$ 

To illustrate the a fore mentioned theory ,some examples of partial differential equations with variable coefficients are discussed in details and the obtained results are exactly the same which is found by variational iteration method.

# **3.**Applications

Here, the extended differential transformation method (DTM) is used to find the solutions of the PDEs in one, two and three dimensions with variable coefficients, and compared with that obtained by other methods.

#### Example 1

Consider the homogeneous Goursat problem

$$u_{xt}=nu$$
 (7)

And the initial condition

 $u(x,0)=U_0=e^x$ 

$$u(0,t)=e^{t},u(0,0)=1$$
 (8)

The form is

$$(k+1)\frac{d}{dx}U_{k+1} = \frac{d}{dx}nU_K \tag{9}$$

If k=0

$$\frac{d}{dx}U_1 = nU_0$$
$$\frac{d}{dx}U_1 = ne^x$$

On integrating, we get

$$U_1 = ne^x \tag{10}$$

If k=1



In general,

$$U_{k} = \frac{n^{k}}{k!} e^{x}$$
(13)

# Example 2:

Consider the homogeneous Goursat problem

$$u_{xt} = u_x$$
 (14)

And the initial condition

$$u(x,0)=U_0=e^x$$
  
 $u(x,0)=e^t, u(0,0)=1$  (15)

The form is,



 $3\frac{d}{dx}U_3 = \frac{d}{dx}U_2$ 

On integrating we get

3U<sub>3</sub>=U<sub>2</sub>

$$3U_3 = \frac{e^x}{2}$$
$$U_3 = \frac{e^x}{3!} \tag{19}$$

In general,

$$U_{K} = \frac{e^{x}}{k!}$$
(20)

#### Example 3:

Consider the homogeneous Goursat problem



If k=1,

$$2\frac{d}{dx}U_2 = \frac{d}{dx}U_1$$

On integrating we get

$$2U_2=n(n)e^{x}$$

(26)

(27)

(28)

(30)

$$U_2 = n^2 \frac{e^x}{2} \tag{25}$$

If k=2,

$$3\frac{d}{dx}U_3 = \frac{d}{dx}nU_2$$

On integrating we get

 $3U_3=n.n^2\frac{e^x}{2}$ 

 $U_3 = n^3 \frac{e^x}{3!}$ 

 $3U_3=nU_2$ 

In general,

 $U_{\rm K}=n^k \frac{e^x}{k!}$ 

## Example 4:

Consider the homogeneous Goursat problem

 $u_{xt} = -2u_x$ 

And the initial condition

 $u(x,0)=U_0=e^x$  $u(0,t)=e^t,u(0,0)=1$ 

The form is,

$$(k+1)\frac{d}{dx}U_{k+1} = \frac{d}{dx}(-2U_k)$$

If k=0,

$$\frac{d}{dx}U_1 = \frac{d}{dx}(-2U_0)$$

On integrating we get,

$$U_1 = -2U_0$$
  
 $U_1 = -2e^x$  (31)



If k=1,

$$2\frac{d}{dx}U_2 = \frac{d}{dx}(-2U_1)$$

On integrating we get

$$2U_{2}=-2U_{1}$$

$$2U_{2}=-2(-2e^{x})$$

$$U_{2}=4\frac{e^{x}}{2}$$
(32)

If k=2,



And the initial condition

$$u(x,0)=U_0=e^x$$
  
 $u(0,t)=e^t,u(0,0)=1$  (36)

The form is,

$$(k+1)\frac{d}{dx}U_{k+1} = \frac{d}{dx}3u_k \tag{37}$$

If k=0

$$\frac{d}{dx}U_1 = \frac{d}{dx}3U_0$$

On integrating we get,

$$U_1=3U_0$$
  
 $U_1=3e^x$  (38)

If k=1,

$$2\frac{d}{dx}U_2 = \frac{d}{dx}3U_1$$

On integrating we get



In general,

$$U_{\rm K}=(3)^k \frac{e^x}{k!} \tag{41}$$

## Example 6:

Consider the homogeneous Goursat problem

 $u_{xt} = -2u_x + 3u \tag{42}$ 

And the initial condition

 $u(x,0)=U_0=e^x$ 

$$u(0,t)=e^{t},u(0,0)=1$$
 (43)

The form is,

$$(k+1)\frac{d}{dx}U_{k+1} = \frac{d}{dx}(-2U_k) + \frac{d}{dx}(3U_k)$$
(44)

If k=0

 $\frac{d}{dx}\mathbf{U}_1 = \frac{d}{dx}(-2\mathbf{U}_0) + \frac{d}{dx}(3\mathbf{U}_0)$ 

On integrating we get,



If k=2,

$$3\frac{d}{dx}U_3 = \frac{d}{dx}(-2U_2) + \frac{d}{dx}(3U_2)$$

On integrating we get

$$3U_3 = -2U_2 + 3U_2$$
$$3U_3 = U_2$$
$$3U_3 = \frac{e^x}{2}$$

$$U_3 = \frac{e^x}{3!} \tag{47}$$

In general,

$$U_{\rm K} = \frac{e^x}{k!} \tag{48}$$

#### Example 7:

Consider the homogeneous Goursat problem



If k=1,

$$2\frac{d}{dx}U_2 = \frac{d}{dx}(mU_1) + \frac{d}{dx}nU_1$$

On integrating we get

If k=2,

On integrating

$$2U_{2}=mU_{1}+nU_{1}$$

$$2U_{2}=m(me^{x}+ne^{x})+n(me^{x}+ne^{x})$$

$$2U_{2}=m^{2}e^{x}+mn e^{x}+mn e^{x}+n^{2}e^{x}$$

$$2U_{2}=m^{2}e^{x}+2mn e^{x}+n^{2}e^{x}$$

$$2U_{2}=(m+n)^{2}e^{x}$$

$$U_{2}=(m+n)^{2}\frac{e^{x}}{2}$$

$$(53)$$

$$3\frac{d}{dx}U_{3}=\frac{d}{dx}(mU_{2})+\frac{d}{dx}(nU_{2})$$
we get



# 4. Conclusion

The differential transform method has been successfully applied for solving partial differential equations with variable coefficients. The solution obtained by differential transform method is an infinite power series for approximate initial condition, which can in turn express the exact solution in a closed form. The result shows that the differential transform method is a powerful mathematical tool for solving partial differential equations with variable coefficients. The reliability of the differential transform method and the reduction in the size of computational domain give this method a wider applicability. Thus, we conclude that the proposed method can be extended to solve many PDEs with variable coefficients which arise in physical and engineering application

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