LINEAR PROGRAMMING PROBLEM USED IN TRANSPORTATION PROBLEM

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Abstract: Mathematics is the queen of science in our daily life, planning is required on various occasions, especially when the resources are limited. Any planning is meant for attaining certain objectives the best strategy is one that gives a maximum output from a minimum input. Linear programming problem used in transportation problem. The Transportation is the movement of people and goods, from one place to another the term is derived from the Latin trans (“across”) and portare (“to carry”) industries. which have the business of providing equipment, actual transport goods and services used in transport of goods or people make up a large broad and important sector of most national economic and are collectively referred to as transport industries.

Keywords: Lpp, Simplex Method, graphical method, transportation problem

I. Introduction

Operations Research is a relatively new discipline. The contents and the boundaries of the OR are not yet fixed. Therefore, to give a formal definition of the term Operations Research is a difficult task. The OR starts when mathematical and quantitative techniques are used to substantiate the decision being taken. The main activity of a manager is the decision making. In our daily life we make the decision seven without noticing them. The decisions are taken simply by common sense, judgment and expertise without using any mathematical or any other model in simple situations. But the decision we are concerned here with are complex and heavily responsible. Examples are public transportation network planning in a city having its own layout of factories, residential blocks or finding the appropriate product mix when there exists a large number of products with different profit contributions and production requirement etc.

Operations Research tools are not from any one discipline. Operations Research takes tools from different discipline such as mathematics, statistics, economics, psychology, engineering etc and combines these tools to make a new set of knowledge for decision making. Today, O.R. became a professional discipline which deals with the application of scientific methods for making decision, and especially to the allocation of scarce resources. The main purpose of O.R. is to provide a rational basis for decisions making in the absence of complete information, because the systems composed of human, machine, and procedures may do not have complete information.

Operations Research can also be treated as science in the sense it describing, understanding and predicting the systems behaviour, especially man-machine system. Thus O.R. specialists are involved in three classical aspect of science, they are as follows: i) Determining the systems behaviour. ii) Analyzing the systems behaviour by developing appropriate models, iii) Predict the future behaviour using these models. The emphasis on analysis of operations as a whole distinguishes the O.R. from other research and engineering. O.R. is an interdisciplinary discipline which provided solutions to problems of military operations during World War II, and also successful in other operations. Today business applications are primarily concerned with O.R. analysis for the possible alternative actions. The business and industry befitted from O.R. in the areas of inventory, reorder policies, optimum location and size of warehouses, advertising policies, etc.

1.1 Definition of operation research:
- Morse and Kimball have stressed O.R. is a quantitative approach and described it as “a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control”.
- Saaty considers O.R. as tool of improving quality of answers. He says, “O.R. is the art of giving bad answers to problems which otherwise have worse answers”.
- Miller and Starr state, “O.R. is applied decision theory, which uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough-going rationality in dealing with his decision problem”.

II. Linear Programming: Linear programming is a powerful quantitative technique (or operational research technique) designs to solve allocation problem. The term ‘linear programming’ consists of the two words ‘Linear’ and ‘Programming’. The word ‘Linear’ is used to describe the relationship between decision variables, which are directly proportional. For example, if doubling (or tripling) the production of a product will exactly double (or triple) the profit and required resources, then it is linear relationship. The word ‘programming’ means planning of activities in a manner that achieves some ‘optimal’ result with available resources. A programme is
‘optimal’ if it maximises or minimises some measure or criterion of effectiveness such as profit, contribution (i.e. sales-variable cost), sales, and cost. Thus, ‘Linear Programming’ indicates the planning of decision variables, which are directly proportional, to achieve the ‘optimal’ result considering the limitations within which the problem is to be solved.

2.1: Step Involved in the Formulation of LP Problem

The steps involved in the formation of linear programming problem are as follows:

Step 1: Identify the Decision Variables of interest to the decision maker and express them as \( X_1, X_2, X_3 \)

Step 2: Ascertain the Objective of the decision maker whether the wants to minimize or to maximize.

Step 3: Ascertain the cost (in case of minimization problem) or the profit (in case of maximization problem) per unit of each of the decision variables.

Step 4: Ascertain the constraints representing the maximum availability or minimum commitment or equality and represent them as less than or equal to (<) type inequality or greater than or equal to (>) type inequality or ‘equal to’ (=) type equality respectively.

Step 5: Put non-negativity restriction as under:

\[ X_j \geq 0; j = 1, 2… n \] (non-negativity restriction)

Step 6: Now formulate the LP problem as under: Subject Maximize (or Minimize)

\[ \text{Figure 2.1: linear program in a standard form} \]

- **Objective Function:** is a linear function of the decision variables representing the objective of the manager/ decision maker.
- **Constraints:** are the linear equations or inequalities arising out of practical limitations.
- **Decision Variables:** are some physical quantities whose values indicate the solution.
- **Feasible Solution:** is a solution which satisfies all the constraints (including the non-negative) presents in the problem.
- **Feasible Region:** is the collection of feasible solutions.
- **Multiple Solutions:** are solutions each of which maximize or minimize the objective function.
- **Unbounded Solution:** is a solution whose objective function is infinite.
- **Infeasible Solution:** means no feasible solution.

**Example 2.1:** A toy company manufactures two types of doll, a basic version-doll A and a deluxe version-doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs. 3.00 and Rs. 5.00 per doll, respectively on doll A and B, then how many of each dolls Should be produced per day in order to maximize the total profit. Formulate this problem.

**Formulation:** Let \( X_1 \) and \( X_2 \) be the number of dolls produced per day of type A and B, respectively. Let the doll A require \( t \) hrs so that the doll B require \( 2t \) hrs. So the total time to manufacture \( X_1 \) and \( X_2 \) dolls should not exceed 2000t hrs. Therefore, \( tX_1 + 2tX_2 < 2000t \), other constraints are simple. Then the linear programming problem become

Maximise \[ p=3X_1+5X_2 \]

Subject to the restriction
\[ X_1+2X_2 \leq 2000 \] (time constraint)
\[ X_1+X_2 \leq 1500 \] (plastics constraints)
\[ X_2 \leq 600 \] (dress constraints)

And non negative restrictions
\[ X_1 \geq 0, X_2 \geq 0 \]

2.12. Graphical method:

**Example 2.2:** solve the given problem by using graphical methods.

Max \( Z = 3x_1 + 2x_2 \)
\[ x_1 + x_2 \leq 80 \]
\[ 2x_1 + x_2 \leq 100 \]
\[ x_1 \leq 40 \]
\[ x_1, x_2 \geq 0 \]

Find the feasible region.
- Plot each constraint as an equation _ line in the plane
- Feasible points on one side of the line – plug in (0,0) to find out which
A corner (extreme) point $X$ of the region $R = \text{every line through } X \text{ intersects } R \text{ in a segments whose one endpoint is } X$. solving a linear program amounts to finding a best corner point by the following theorem.

**III. Simplex method**

The process consists of two steps

1. Find a feasible solution (or determine that none exists).
2. Improve the feasible solution to an optimal solution.

**3.1 Canonical form:**
Linear program (LP) is in a canonical form if

- all constraints are equations.
- all variables are non-negative.

\[
\begin{align*}
\text{max} & \quad c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{subject to} & \quad a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\
& \quad a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\
& \quad \vdots \\
& \quad a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m \\
& \quad x_1, x_2, \ldots, x_n \geq 0
\end{align*}
\]

**3.2 Slack variables:** To change an inequality to an equation, we add a new non-negative variable called a slack variable.

\[
\begin{align*}
x_1 + x_2 & \leq 80 \quad \rightarrow \quad x_1 + x_2 + s_1 = 80 \\
x_1 + x_2 & \geq 80 \quad \rightarrow \quad x_1 + x_2 - e_1 = 80
\end{align*}
\]
Notes:
- the variable $e_1$ is sometimes called an excess variable.
- We can use $x_i$ for slack variables (where $i$ is a new index)

### 3.3 Simplex method

**Example: 3.31** Consider the 2.2 example toyshop example above. Convert to equalities by adding slack variables.

\[
\begin{align*}
\text{max } & 3x_1 + 2x_2 \\
& x_1 + x_2 \leq 80 \\
& 2x_1 + x_2 \leq 100 \\
& x_1 \leq 40 \\
& x_1, x_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max } & 3x_1 + 2x_2 + x_3 = 80 \\
& 2x_1 + x_2 + x_4 = 100 \\
& x_1 + x_5 = 40 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{align*}
\]

*Figure 3.31 graphical representation of simplex method*

**Step: Starting feasible solution**
- Set variables $x_1, x_2$ to zero and set slack variables to the values on the right-hand side yields a feasible solution $x_1 = x_2 = 0, x_3 = 80, x_4 = 100, x_5 = 40$
- Recall that the solution is feasible because all variables are non-negative and satisfy all equations (we get a feasible solution right away because the right-hand side is non-negative; this may not always work)
- Note something interesting: in this feasible solution two variables (namely $x_1, x_2$) are zero. Such a solution is called a basic solution of this problem, because the value of at least two variables is zero.
- In a problem with $n$ variables and $m$ constraints, a solution where at least $(n - m)$ variables are zero is a basic solution. A basic solution that is also feasible is called a basic feasible solution (BFS). The importance of basic solutions is revealed by the following observation.
- Basic solutions are precisely the corner points of the feasible region. Recall that we have discussed that to find an optimal solution to an LP, it suffices to find a best solution among all corner points. The above tells us how to compute them. They are the basic feasible solutions.
- A variable in a basic solution is called a non-basic variable if it is chosen to be zero. Otherwise, the variable is basic.
- The basic variables we collectively call a basis.

### 3.4. Dictionary

To conveniently deal with basic solutions, we use the so-called dictionary. A dictionary lists values of basic variables as a function of non-basic variables. The correspondence is obtained by expressing the basic variables from the initial set of equations. (We shall come back to this later; for now, have a look below.)

Express the slack variables from the individual equations.

\[
\begin{align*}
\text{max } & 3x_1 + 2x_2 + x_3 = 80 \\
& 2x_1 + x_2 + x_4 = 100 \\
& x_1 + x_5 = 40 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{align*}
\]

\[
\begin{align*}
x_3 &= 80 - x_1 - x_2 \\
x_4 &= 100 - 2x_1 - x_2 \\
x_5 &= 40 - x_1 \\
z &= 0 + 3x_1 + 2x_2
\end{align*}
\]

$x_1, x_2$ independent (non-basic) variables

$x_3, x_4, x_5$ dependent (basic) variables

$\{x_3, x_4, x_5\}$ is a basis set $x_1 = x_2 = 0$ the corresponding (feasible) solution is $x_3 = 80, x_4 = 100, x_5 = 40$ with value $z = 0$

**Improving the solution:**

Try to increase $x_1$ from its current value 0 in hopes of improving the value of $x_1 = 20, x_2 = 0$ and substitute into the dictionary to obtain the values of $x_3, x_4, x_5$ and $Z = x_3 = 60, x_4 = 0, x_5 = 20$ with the value of $x = 60 \rightarrow$ feasible

Try again $x_1 = 40, x_2 = 0 \rightarrow x_3 = 40, x_4 = 20, x_5 = 0$ with the value $z = 120 \rightarrow$ feasible.
Now try $x_1=50, x_2=0 \rightarrow x_3=30, x_4=0, x_5=10 \rightarrow$ not feasible.

If $x_1 = t$ and $x_2 = 0$, then the solution is feasible if

\[
\begin{align*}
  x_3 & = 80 - t - 0 \geq 0 \\
  x_4 & = 100 - 2t - 0 \geq 0 \\
  x_5 & = 40 - t \geq 0
\end{align*}
\implies \begin{cases} t \leq 80 \\ t \leq 50 \\ t \leq 40 \end{cases} \implies t \leq 40
\]

Maximal value is $x_1 = 40$ at which point the variable $x_3$ becomes zero $x_1$ is incoming variable and $x_5$ is outgoing variable (we say that $x_1$ enters the dictionary/basis, and $x_5$ leaves the dictionary/basis)

**Ratio test**

The above analysis can be streamlined into the following simple “ratio” test

\[
\begin{align*}
  x_3 & \overset{80}{1} = 80 \\
  x_4 & \overset{100}{2} = 50 \\
  x_5 & \overset{40}{1} = 40
\end{align*}
\]

(watch-out: we only consider this ratio because the coefficient of $x_1$ is negative ($-2$)...more on that in the later steps)

Minimum achieved with $x_5 = 0$ outgoing variable

Express $x_1$ from the equation for $x_5$

\[
x_5 = 40 - x_1
\]

Substitute $x_1$ to all other equations new feasible dictionary

\[
\begin{align*}
  x_1 & = 40 - x_5 \\
  x_3 & = 80 - (40 - x_5) - x_2 \\
  x_4 & = 100 - 2(40 - x_5) - x_2 \\
  z & = 0 + 3(40 - x_5) + 2x_2
\end{align*}
\]

now $x_2, x_5$ are independent variables and $x_1, x_3, x_4$ are dependent $\rightarrow \{x_1, x_3, x_4\}$ is a basis.

we repeat: we increase $x_2 \rightarrow$ incoming variable, ratio test:

\[
x_1 : \text{does not contain } x_2 \rightarrow \text{no constraint}
\]

Minimum achieved for $x_4 \rightarrow$ outgoing variable

\[
x_4 = 20 + x_2 + 2x_5
\]

\[
\begin{align*}
  x_1 & = 40 - x_5 \\
  x_3 & = 40 - (20 - x_4 + 2x_5) - x_5 \\
  x_4 & = 20 - x_4 + 2x_5 \\
  z & = 120 + 2(20 - x_4 + 2x_5) - 3x_5
\end{align*}
\]

$X_5$ incoming variable, ratio test

\[
x_1 : \overset{40}{1} = 40
\]

$X_2$: positive coefficient $\rightarrow$ no constant

\[
x_3 : \overset{20}{1} = 20
\]

Minimum achieved for $x_3 \rightarrow$ outgoing variable

\[
x_3 = 20 + x_4 + x_5
\]

\[
\begin{align*}
  x_1 & = 40 - x_5 \\
  x_2 & = 20 - x_4 + x_5 \\
  x_3 & = 20 - x_4 + x_5 \\
  z & = 160 - 2x_4 + x_5
\end{align*}
\]
No more improved possible → optimal solution.

\[ x_1 = 20, x_2 = 60, x_3 = 0, x_4 = 0 \text{ of the value } z = 180 \]

Why setting \( x_3, x_4 \) to any non-zero values results in a smaller value of \( z \).

Each dictionary is equivalent to the original system (the two have the same set of solutions).

### 3.5 Simplex algorithm:

**Preparation:** find a starting feasible solution/dictionary

1. Convert to the canonical form (constraints are equalities) by adding slack variables \( x_{n+1}, \ldots, x_{n+m} \)
2. Construct a starting dictionary - express slack variables and objective function \( z \)
3. If the resulting dictionary is feasible, then we are done with preparation
   - If not, try to find a feasible dictionary using the Phase I method.

### 3.6 Simplex step (maximization LP): try to improve the solution

1. **(Optimality test):** If no variable appears with a positive coefficient in the equation for \( z \rightarrow \) STOP, current solution is optimal
   - set non-basic variables to zero
   - read off the values of the basic variables and the objective function \( z \rightarrow \) Hint: the values are the constant terms in respective equations
   - report this (optimal) solution
2. Else pick a variable \( x_i \) having positive coefficient in the equation for \( z \)
3. **Ratio test:** in the dictionary, find an equation for a variable \( x_j \) in which
   - \( x_i \) appears with a negative coefficient \(-a\)
   - the ratio \( \frac{b}{a} \) is smallest possible (where \( b \) is the constant term in the equation for \( x_j \))
4. If no such \( x_j \) exists \( \rightarrow \) stop, no optimal solution, report that LP is unbounded
5. Else \( x_j \equiv \) outgoing variable → construct a new dictionary by pivoting:
   - express \( x_i \) from the equation for \( x_j \)
   - add this as a new equation,
   - remove the equation for \( x_j \),
   - substitute \( x_i \) to all other equations (including the one for \( z \))
6. Repeat from 1.

### 3.7 Two phase Simplex method

Canonical form = equations, non-negative variables

\( n = \) number of variables

\( m = \) number of equations

basic solution = at least \( (n - m) \) variables are zero

basic solutions = dictionaries

basic feasible solutions = corner/extreme points = feasible dictionaries

**Example:**

\[
\begin{align*}
\max \quad & 3x_1 + 2x_2 \\
x_1 + x_2 & \leq 80 \\
2x_1 + x_2 & \leq 100 \\
x_1 & \leq 40 \\
x_1, x_2 & \geq 0
\end{align*}
\]

**Feasible dictionary:**

\[
\begin{align*}
x_1 &= 20 + x_3 - x_4 \\
x_2 &= 60 - 2x_3 + x_4 \\
x_5 &= 20 - x_3 + x_4 \\
z &= 180 - x_3 - x_4
\end{align*}
\]
A special class of linear programming problem is the Transportation Problem, where the objective is to minimize the cost of distributing a product from a number of sources (e.g., factories) to a number of destinations (e.g., warehouses) while satisfying both the supply limits and the demand requirement. Because of the special structure of the Transportation Problem, the Simplex Method of solving is unsuitable for this problem. The model assumes that the distributing cost on a given route is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment. The transportation problem's special feature is illustrated here with the help of the following Example 1.1.

**Example 4.1:**
Suppose a manufacturing company owns three factories (sources) and distributes his products to five different retail agencies (destinations). The following table shows the capacities of the three factories, the quantity of products required by the various retail agencies, and the cost of shipping one unit of the product from each of the three factories to each of the five retail agencies.

<table>
<thead>
<tr>
<th>Retail Agency</th>
<th>Factories</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>100</td>
<td>60</td>
<td>50</td>
<td>50</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>100</td>
<td>60</td>
<td>50</td>
<td>50</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>100</td>
<td>60</td>
<td>50</td>
<td>50</td>
<td>100</td>
<td>50</td>
</tr>
</tbody>
</table>

Usually, the above table is referred to as a Transportation Table, which provides basic information regarding the transportation problem. The quantities inside the table are known as transportation cost per unit of product. The capacity of the factories 1, 2, 3 is 50, 100, and 150 respectively. The requirement of the retail agency 1, 2, 3, 4, 5 is 100, 60, 50, 50, and 40 respectively.

In this case, the transportation cost of one unit
- from factory 1 to retail agency 1 is 1,
- from factory 1 to retail agency 2 is 9,
- from factory 1 to retail agency 3 is 13, and so on.

A transportation problem can be formulated as a linear programming problem using variables with two subscripts.

Let $X_{11} = \text{Amount to be transported from factory 1 to retail agency 1}$

$X_{12} = \text{Amount to be transported from factory 1 to retail agency 2}$

$\ldots$ $\ldots$

$\ldots$ $\ldots$

$\ldots$ $\ldots$

$X_{35} = \text{Amount to be transported from factory 3 to retail agency 5}$.

Let the transportation cost per unit be represented by $C_{11}$, $C_{12}$, $\ldots$, $C_{35}$ that is $C_{11} = 1$, $C_{12} = 9$, and so on.

Let the capacities of the three factories be represented by $a_1 = 50$, $a_2 = 100$, $a_3 = 150$.

Let the requirement of the retail agencies be $b_1 = 100$, $b_2 = 60$, $b_3 = 50$, $b_4 = 50$, and $b_5 = 40$.

Thus, the problem can be formulated as

Minimize

$$\max 3x_1 + 2x_2$$

$$x_1 + x_2 + x_3 = 80$$

$$2x_1 + x_2 + x_4 = 100$$

$$x_1 + x_5 = 40$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Infeasible dictionary: $x_3 - x_5 = 0$

$$x_1 = 40 - x_5$$

$$x_2 = 40 - x_3 + x_5$$

$$x_4 = -20 + x_3 + x_5$$

$$z = 200 - 2x_3 - x_5$$

(it is infeasible since $x_4 = -20$)
Thus, the problem has 8 constraints and 15 variables. So, it is not possible to solve such a problem using simplex method. This is the reason for the need of special computational procedure to solve transportation problem. There are varieties of procedures, which are described in the next section.

4.1 Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

**Step 1:** Determine a starting basic feasible solution, using any one of the following three methods
1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

**Step 2:** Determine the optimal solution using the following method
MODI (Modified Distribution Method) or UV Method.

**Basic Feasible Solution of a Transportation Problem**

The computation of an initial feasible solution is illustrated in this section with the help of the Example 4.1 discussed in the previous section. The problem in the example 1.1 has 8 constraints and 15 variables we can eliminate one of the constraints since \( \sum a_i = \sum b_j \). Thus now the problem contains 7 constraints and 15 variables. Note that any initial (basic) feasible solution has at most \( m + n - 1 \) non-zero \( X_{ij} \). Generally, any basic feasible solution with \( m \) sources (such as factories) and \( n \) destination (such as retail agency) has at most \( m + n - 1 \) non-zero \( X_{ij} \). The special structure of the transportation problem allows securing a non artificial basic feasible solution using one the following three methods.

- North West Corner Method
- Least Cost Method
- Vogel Approximation Method

The difference among these three methods is the quality of the initial basic feasible solution they produce, in the sense that a better initial solution yields a smaller objective value. Generally the Vogel Approximation Method produces the best initial basic feasible solution, and the North West Corner Method produces the worst, but the North West Corner Method involves least computations.

4.2 North West Corner Method

The method starts at the North West (upper left) corner cell of the tableau (variable \( x_{11} \)).

**Step 1:** Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

**Step 2:** Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column becomes zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed out row (column).

**Step 3:** If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step 1.

**Example 4.2:**

Consider the problem discussed in Example 4.1 to illustrate the North West Corner Method of determining basic feasible solution:

<table>
<thead>
<tr>
<th>Retail Agency</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factors</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>9</td>
<td>13</td>
<td>36</td>
<td>51</td>
<td>50</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>33</td>
<td>1</td>
<td>23</td>
<td>26</td>
<td>150</td>
</tr>
<tr>
<td>Requirement</td>
<td>100</td>
<td>60</td>
<td>50</td>
<td>50</td>
<td>40</td>
<td>300</td>
</tr>
</tbody>
</table>

The allocation is shown in the following tableau:
The arrows show the order in which the allocated (bolded) amounts are generated. The starting basic solution is given as

\[ x_{11} = 50, \]
\[ x_{21} = 50, x_{22} = 50, \]
\[ x_{23} = 10, x_{33} = 50, x_{34} = 50, x_{35} = 40. \]

The corresponding transportation cost is
\[ 50 \times 1 + 50 \times 24 + 50 \times 12 + 10 \times 33 + 50 \times 1 + 50 \times 23 + 40 \times 26 = 4420. \]

It is clear that as soon as a value of \( X_{ij} \) is determined, a row (column) is eliminated from further consideration. The last value of \( X_{ij} \) eliminates both a row and column. Hence a feasible solution computed by North West Corner Method can have at most \( m + n - 1 \) positive \( X_{ij} \) if the transportation problem has \( m \) sources and \( n \) destinations.

### 4.3 Least Cost Method

The least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which \( C_{ij} \) is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column. We then cross out the satisfied row or column, and adjust the amounts of capacity and requirement accordingly. If both a row and a column is satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until we are left at the end with exactly one uncrossed-out row or column.

**Example 4.4:**

The least cost method of determining initial basic feasible solution is illustrated with the help of problem presented in the section 4.1.

The least cost method is applied in the following manner:

We observe that \( C_{11}=1 \) is the minimum unit cost in the table. Hence \( X_{11}=50 \) and the first row is crossed out since the row has no more capacity. Then the minimum unit cost in the uncrossed-out row and column is \( C_{23}=1 \), hence \( X_{23}=40 \) and the fifth column is crossed out. Next \( C_{33}=1 \) is the minimum unit cost, hence \( X_{33}=50 \) and the third column is crossed out. Next \( C_{22}=12 \) is the minimum unit cost, hence \( X_{22}=60 \) and the second column is crossed out. Next we look for the uncrossed-out row and column now \( C_{31}=14 \) is the minimum unit cost, hence \( X_{31}=50 \) and crossed out the first column since it was satisfied. Finally \( C_{34}=23 \) is the minimum unit cost, hence \( X_{34}=50 \) and the fourth column is crossed out. So that the basic feasible solution developed by the Least Cost Method has transportation cost is
\[ 1 \times 50 + 12 \times 60 + 1 \times 40 + 14 \times 50 + 1 \times 50 + 23 \times 50 = 2710. \]

Note that the minimum transportation cost obtained by the least cost method is much lower than the corresponding cost of the solution developed by using the north-west corner method.

### 4.4 Vogel Approximation Method (VAM):

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

**Step 1:** For each row (column) with strictly positive capacity (requirement), determine a penalty by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column).

**Step 2:** Identify the row or column with the largest penalty among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column.

**Step 3:** We select \( X_{ij} \) as a basic variable if \( C_{ij} \) is the minimum cost in the row or column with largest penalty. We choose the numerical value of \( X_{ij} \) as high as possible subject to the row and the column constraints. Depending upon whether \( a_i \) or \( b_j \) is the smaller of the two \( i \)th row or \( j \)th column is crossed out.

**Step 4:** The Step 2 is now performed on the uncrossed-out rows and columns until all the basic variables have been satisfied.

**Example 4.5:**

Consider the following transportation problem

<table>
<thead>
<tr>
<th>Origin</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>( s_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>32</td>
<td>17</td>
<td>4</td>
<td>120</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>37</td>
<td>9</td>
<td>7</td>
<td>70</td>
</tr>
<tr>
<td>3</td>
<td>32</td>
<td>37</td>
<td>20</td>
<td>15</td>
<td>50</td>
</tr>
<tr>
<td>( b_j )</td>
<td>60</td>
<td>40</td>
<td>30</td>
<td>110</td>
<td>240</td>
</tr>
</tbody>
</table>
Note: $a_i$ = capacity (supply),$b_j$ = requirement (demand)

Now, compute the penalty for various rows and columns which is shown in the following table:

<table>
<thead>
<tr>
<th>Origin</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Column Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>29</td>
<td>32</td>
<td>20</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>35</td>
<td>9</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>34</td>
<td>37</td>
<td>19</td>
<td>33</td>
<td>110</td>
</tr>
<tr>
<td>$b_j$</td>
<td>60</td>
<td>40</td>
<td>30</td>
<td>140</td>
<td>240</td>
</tr>
<tr>
<td>Row Penalty</td>
<td>4</td>
<td>15</td>
<td>8</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Look for the highest penalty in the row or column, the highest penalty occurs in the second column and the minimum unit cost $c_{ij}$ in this column is $c_{12}$=22. Hence assign 40 to this cell i.e. $x_{12}$=40 and cross out the second column (since second column was satisfied. This is shown in the following table:

<table>
<thead>
<tr>
<th>Origin</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Column Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>29</td>
<td>32</td>
<td>20</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>35</td>
<td>9</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>34</td>
<td>37</td>
<td>19</td>
<td>33</td>
<td>110</td>
</tr>
<tr>
<td>$b_j$</td>
<td>60</td>
<td>40</td>
<td>30</td>
<td>140</td>
<td>240</td>
</tr>
<tr>
<td>Row Penalty</td>
<td>4</td>
<td>15</td>
<td>8</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

The next highest penalty in the uncrossed-out rows and columns is 13 which occur in the first row and the minimum unit cost in this row is $c_{14}$=4, hence $x_{14}$=80 and cross out the first row. The modified table is as follows:

<table>
<thead>
<tr>
<th>Origin</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Column Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>24</td>
<td>35</td>
<td>9</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>34</td>
<td>37</td>
<td>19</td>
<td>33</td>
<td>110</td>
</tr>
<tr>
<td>$b_j$</td>
<td>60</td>
<td>40</td>
<td>30</td>
<td>140</td>
<td>240</td>
</tr>
<tr>
<td>Row Penalty</td>
<td>4</td>
<td>15</td>
<td>8</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

The next highest penalty in the uncrossed-out rows and columns is 8 which occurs in the third column and the minimum cost in this column is $c_{33}$=9, hence $x_{33}$=30 and cross out the third column with adjusted capacity, requirement and penalty values. The modified table is as follows:

<table>
<thead>
<tr>
<th>Origin</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Column Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>24</td>
<td>35</td>
<td>9</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>34</td>
<td>37</td>
<td>19</td>
<td>33</td>
<td>110</td>
</tr>
<tr>
<td>$b_j$</td>
<td>60</td>
<td>40</td>
<td>30</td>
<td>140</td>
<td>240</td>
</tr>
<tr>
<td>Row Penalty</td>
<td>8</td>
<td>15</td>
<td>8</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the second row and the smallest cost in this row is $c_{24}$=15, hence $x_{24}$=30 and cross out the fourth column with the adjusted capacity, requirement and penalty values. The modified table is as follows:

<table>
<thead>
<tr>
<th>Origin</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Column Penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>24</td>
<td>35</td>
<td>9</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>34</td>
<td>37</td>
<td>19</td>
<td>33</td>
<td>110</td>
</tr>
<tr>
<td>$b_j$</td>
<td>60</td>
<td>40</td>
<td>30</td>
<td>140</td>
<td>240</td>
</tr>
<tr>
<td>Row Penalty</td>
<td>8</td>
<td>15</td>
<td>8</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

The transportation cost corresponding to this choice of basic variables is

$$22 \times 40 + 4 \times 80 + 9 \times 30 + 7 \times 30 + 24 \times 30 + 10 \times 10 + 32 \times 50 = 3520$$
V. CONCLUSION

Linear programming problem is like a vast ocean where many methods, advantages, uses, requirement etc can seen. Linear programming can be done in any sector where there is less waste and more profit. By this the production of anything is possible through the new method of L.P. Linear programming is a goods technique, especially in the business sector.

Transportation Problem is a special kind of linear programming problem. Because of the transportation problem special structure the simplex method is not suitable. But which may be utilized to make efficient computational techniques for its solution. Generally transportation problem has a number of origins and destination. A certain amount of consignment is available in each origin. Similarly, each destination has a certain demand/requirements. The transportation problem represents amount of consignment to be transported from different origins to destinations so that the transportation cost is minimized without violating the supply and demand constraints.

There are two phases in the transportation problem. First is the determination of basic feasible solution and second is the determination of optimum solution. There are three methods available to determine the basic feasible solution, they are: 1. North West Corner Method 2. Least Cost Method or Matrix Minimum Method 3. Vogel’s Approximation Method (VAM)

Transportation problem can be generalized into a Transshipment Problem where transportation of consignment is possible from origin to origin or destination as well as destination to origin or destination. The Transshipment problem may be result in an economy way of shipping in some situations.

References