ON DOUBLE REPRESENTATION OF QUATERNION QUASI-NORMAL MATRICES

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ABSTRACT

In this paper, the properties of quaternion quasi-normal matrices in the form of double representation of complex matrices. The normal product of the quaternion quasi-normal matrices are derived.

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INTRODUCTION:

A normal matrix $A = (a_{ij})$ with complex elements is a matrix such that $AA^T = A^T A$ where $A^T$ denotes the (complex) conjugate transpose of $A$. In an article by K. Morita[5] a quasi-normal matrix is defined to be a complex matrix $A$ which is such that $AA^C = A^T A^C$, where $T$ denotes the transpose of $A$ and $A^C$ the matrix in which each element is replaced by its conjugate, and certain basic properties of such a matrix are developed there.


In this paper, quaternion quasi-normal matrix is defined. The further properties of quaternion quasi-normal are developed, their relation in a sense, to a quaternion normal matrices are consider and further results concerning quaternion normal products are obtained for quaternion quasi-normal.

Theorem: 1

A matrix $A$ is double representation of quaternion quasi-normal iff a quaternion unitary matrix $U$ such that $UAU^T$ is a direct sum of non-negative real numbers and of $2 \times 2$ matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a$ and $b$ are non-negative real numbers.

Proof:

Let $A$ be a double representation of quaternion quasi-normal where $A = X + Y$. [Where $X = X_0 + X_1 j$ and $Y = Y_0 + Y_1 j$]. Where $X = X^T = X_0^T + X_1^T j$ and $Y = -Y^T = -(Y_0^T + Y_1^T j)$. Then $AA^C = A^T A^C$ where $A = X + Y$.

$$AA^C = (X + Y)(X + Y)^C$$

$$= [(X_0 + Y_0) + (X_1 + Y_1) j][(X_0 + X_1) j + (Y_0 + Y_1) j]^C$$

$$= [(X_0 + Y_0) + (X_1 + Y_1) j][(X_0 + X_1) j + (Y_0 + Y_1) j]^C$$

$$= [(X_0 + Y_0) + (X_1 + Y_1) j][(X_0^C - X_1^C j) + (-Y_0^C + Y_1^C j)]$$
\[
= [(X_0 + Y_0) + (X_1 + Y_1)j][(X_0^C - Y_0^C) - (X_1^C - Y_1^C)j]
= [(X_0 + Y_0)(X_0^C - Y_0^C)] - [(X_1 + Y_1)(X_1^C - Y_1^C)j]
= [X_0X_0^C - X_0Y_0^C + Y_0X_0^C - Y_0Y_0^C] + [X_1X_1^C - X_1Y_1^C - Y_1X_1^C + Y_1Y_1^C]j \quad \ldots \quad (1)
\]

Now,
\[
A^TA^C = (X + Y)^T(X + Y)^C
\]
\[
= (X_0 + X_1j + Y_0 + Y_1j)^T(X_0 + X_1j + Y_0 + Y_1j)^C
\]
\[
= [(X_0 + Y_0) + (X_1 + Y_1)j]^T[(X_0 + X_1j) + (Y_0 + Y_1j)]C
\]
\[
= [(X_0^T + Y_0^T) + (X_1^T + Y_1^T)j]((X_0^C + Y_0^C) - (X_1^C + Y_1^C)j]
\]
\[
= [(X_0^T + X_1^T)j + (Y_0^T + Y_1^T)j]((X_0^C + Y_0^C) - (X_1^C + Y_1^C)j]
\]
\[
= [(X_0 - Y_0)(X_0^C + Y_0^C)] - [(X_1 - Y_1)(X_1^C + Y_1^C)j]
\]
\[
= [X_0X_0^C + X_0Y_0^C - Y_0X_0^C - Y_0Y_0^C] + [Y_1X_1^C + Y_1Y_1^C - X_1X_1^C - X_1Y_1^C]j \quad \ldots \quad (2)
\]

Since \( A \) is double representation of quaternion quasi-normal,
\[
AA^{CT} = A^TA^C
\]
\[
[X_0X_0^C - X_0Y_0^C + Y_0X_0^C - Y_0Y_0^C] + [X_1X_1^C - X_1Y_1^C + Y_1X_1^C + Y_1Y_1^C]j = [X_0X_0^C + X_0Y_0^C - Y_0X_0^C - Y_0Y_0^C] + [Y_1X_1^C + Y_1Y_1^C - X_1X_1^C - X_1Y_1^C]j
\]
\[
Y_0X_0^C + Y_0X_0^C + X_1Y_1^C + X_1Y_1^Cj = X_0X_0^C + X_0Y_0^C + Y_1X_1^C + Y_1X_1^Cj = 2X_0Y_0^C + 2X_1Y_1^Cj
\]
\[
Y_0X_0^C - Y_1X_1^Cj = Y_0X_0^C - X_1X_1^Cj
\]
\[
(Y_0 + Y_1j)(X_0^C - X_1^C)j = (X_0 + X_1j)(Y_0^C - Y_1^Cj)
\]
\[
YX^C = XY^C
\]

There exists a quaternion unitary matrix \( U = U_0 + U_1j \) such that \( U_0 + U_1j \)[?], \( UXU^T = (U_0X_0U_0^T) + (U_1X_1U_1^T) = D \) is a diagonal matrix with non-negative real. Therefore,
\[
(UYU^T)(UXU^T)^C = (UXU^T)(UYU^T)^C
\]
\[
U_0Y_0U_0^TU_0^C U_0^TU_0^C U_0^TCU_0^TCU_0^Tj = U_0X_0U_0^TU_0^CU_0^TCU_0^TCU_0^Tj
\]

Or \( WD = DW^C \), where \( W = -W^T \). Let \( U_0 + U_1j \) be chosen so that \( D \) is such that \( d_i \geq d_j \geq 0 \) for \( s < t \) where \( d_i \) is the \( s \th \) diagonal element of \( D \).

If \( W = (e_{st}) \) where \( e_{ts} = -e_{st} \), then \( e_{st}d_i = d_i e_{st} \) for \( t > s \) and three possibilities may occur: if \( d_i = d_j \neq 0 \), the \( e_{st} \) is real; if \( d_s = d_t = 0 \), \( e_{st} \) is arbitrary (though \( W = -W^T \) still holds); and if \( d_s \neq d_t \), then \( e_{st} = 0 \) for if \( e_{st} = a + ib \) then \( (a + ib) d_i = d_i (a - ib) \) and \( a (d_i - d_j) = 0 \) implies that \( a = 0 \) and \( b (d_i + d_j) = 0 \) implies that \( d_i = -d_j \) (which is not possible since \( d_i \) are real and non-negative and \( d_s \neq d_t \) or \( b = 0 \) so \( e_{st} = 0 \).

So if \( UXU^T = (U_0X_0U_0^T) + (U_1X_1U_1^T)j = d_1 I_1 \oplus d_2 I_2 \oplus \ldots \oplus d_k I_k \) where \( \oplus \) denotes the direct sum, then
\[
UYU^T = (U_0Y_0U_0^T) + (U_1Y_1U_1^T)j = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_k \) where \( Y_i = -Y_i^T \) is real and \( Y_k = -Y_k^T \) is quaternion if
and only if \( d_k = 0 \). For each real \( Y_s \) there exists a real orthogonal matrix \( V_s \) so that \( V_s Y_s V_s^T \) is a direct sum of zero matrices and matrices of the form

\[
\begin{bmatrix}
0 & b_1 + b_2 j \\
-b_1 - b_2 j & 0
\end{bmatrix}
\]

where \( b_1 \) and \( b_2 \) are real.

If \( Y_k = (Y_{0(k)} + Y_{1(k)}j) \) is quaternion, there exists a quaternion unitary matrix \( V_k = V_{0(k)} + V_{1(k)}j \) such that \( V_{0(k)} Y_{0(k)} V_{0(k)} + V_{1(k)} Y_{1(k)} V_{1(k)} \), \( Y_0 + Y_j \) is a direct sum of matrices of the same form, so that if \( V = V_1 \otimes V_2 \otimes \ldots \otimes V_k \), then \( V = V_0 + V_1j \), then \( VUXU^TV^T = (V_0 U_0 X_0 U_0^TV_0^T) + (V_1 U_1 X_1 U_1^TV_1^T) \) \( = D \) and \( VUUXU^TV^T = (V_0 U_0 Y_0 U_0^TV_0^T) + (V_1 U_1 Y_1 U_1^TV_1^T) j \) \( = F \) the direct sum described. Therefore, \( VUAU^TV^T = VU(X + Y)U^TV^T \) \( = [(V_0 U_0 X_0 U_0^TV_0^T) + (V_1 U_1 X_1 U_1^TV_1^T) j] + [(V_0 U_0 Y_0 U_0^TV_0^T) + (V_1 U_1 Y_1 U_1^TV_1^T) j] = D + F \) which is the desired form.

Properties of double representation of quaternion quasi-normal matrices:

If \( A_0 + A_1j \) and \( B_0 + B_1j \) are two quaternion quasi-normal matrices such that that \( AB^C = BA^C \), that is \( A_0 B_0^C - A_0 B_1^C j = B_0 A_0^C - B_0 A_1^C j \) then \( A_0 + A_1j \) and \( B_0 + B_1j \) can be simultaneously brought into the above quaternion normal form under the same \( U_0 + U_1j \) (with a generalization to a finite number) but not conversely; if \( (A_0 + A_1j) \) is quaternion quasi-normal \( AA^C = A_0 A_0^C - A_0 A_1^C j \) is quaternion normal in the usual sense, but not conversely; and if \( (A_0 + A_1j) \) is quaternion quasi-normal and \( AA^C \) is real, then there is a real orthogonal matrix which gives the above form.

Properties of double representation of quaternion quasi-normal matrices not obtained in this section but of subsequent use are the following:

a) \( A = A_0 + A_1j \) is both quaternion normal and quaternion quasi-normal matrices if and only if \( A_0 + A_1j = H_0 U_0 + H_1 U_1j \) so \( H = H_0 + H_1j = H_0^T + H_1^T j \) so that \( H \) is real.

b) If \( A_0 + A_1j = H_0 U_0 + H_1 U_1j = U_0^T H_0^T + U_1^T H_1^T j \) is quaternion quasi-normal matrices, then \( U_0 H_0 + U_1 H_1j \) is quaternion quasi-normal matrices, if and only if \( H_0 U_0^2 + H_1 U_1^2 j = U_0^2 H_0 + U_1^2 H_1j \), that is if and only if \( H_0 U_0^2 + H_1 U_1^2 j \) is quaternion normal.

For if \( U_0 H_0 + U_1 H_1j \) is quaternion quasi-normal matrices, \( U_0 H_0 + U_1 H_1j = H_0^T U_0 + H_1^T U_1j \) so that \( H_0 U_0^2 + H_1 U_1^2 j = U_0 H_0^T U_0 + U_1 H_1^T U_1j = U_0^T H_0 + U_1^2 H_1j \) and if \( H_0 U_0^2 + H_1 U_1^2 j = U_0^2 H_0 + U_1^2 H_1j \), then \( HUU \) 

\[
H^T U = H_0^T U_0 + H_1^T U_1j = U_0 H_0 + U_1 H_1j = UH
\]

Theorem: 2

If \( A_0 + A_1j \) and \( B_0 + B_1j \) are quaternion quasi-normal matrices, then \( A_0 B_0 + A_1 B_1j \) is quaternion normal if and only if \( A_0^T A_0 B_0 - A_1^T A_0 B_1j = B_0^T A_0 A_0^T - B_0^T A_1 A_1^T j \) and \( A_0 B_0 B_0^T - A_0 B_1 B_1^T j = B_0^T B_0 A_0 - B_1^T B_1 A_1j \) (that is if and only if each is “quaternion normal relative to the other”).

Proof:
If $A_B^j + A_B^j$ is quaternion normal, from the above, $D = D_0 + D_1j$,
\[
D^CDB_2 = D_0^CDB_0(2) - D_1^DD_1(2)B_2^j = B_2^0D_0^CDB_0(2) - B_2^1D_1(2)B_1(2)D_1^C
\]
so that $F_0^CDB_0(0) - F_1^CDB_1(0)j$ or
\[
A_0^CDB_0(0) - A_1^CDB_1(0)j = B_0^0A_0^CT - B_1^1A_1^CT
\]
Similarly, since $DB_2 = D_0^B_0(2) + D_1^B_1(2)j$ is quaternion normal, \[
DB_2^CDB_2D^C = 
\]
\[
D_0B_0^C(0) + B_1^CDB_2(2)B_2^C(2)j = B_2^0D_0^CDB_0(2) + B_2^1D_1(2)B_1(2)D_1^C
\]
so, $DB_2B_2^CT = D_0B_0^C(0) + B_1^CDB_2(2)B_2^C(2)j = B_2^0D_0^CDB_0(2) - B_2^1D_1(2)B_1(2)j$ or \[
F_0B_0^C(0) - F_1B_1^CDB_2(2)B_2^C(2)j = B_2^0B_0^C(0) - B_2^1B_1^CDB_2(2)B_2^C(2)j = B_2^0B_0^CT - B_2^1B_1^CTB_1j
\]
That is \[
ABB^CT = B^CT BA
\]
The converse is directly verifiable.

**Double Representation Of Quaternion Quasi-Normal Products of Matrices:**

It is possible if $A_B^j + A_B^j$ is quaternion normal and $B_0 + B_1j$ is quaternion quasi-normal that $A_B^j$ is quaternion quasi-normal.

For example,

Any quaternion quasi-normal matrix $C = H_UU_0 + H_UU_1j = U_0H_U^T + U_1H_U^Tj$ is such a product with $A = A_0 + A_1j = H_0 + H_1j$ and $B_0 + B_1j = U_0 + U_1j$ or if $C = H_0U_0 + H_1U_1j = U_0H_0^T + U_1H_1^Tj$ and $A_0 + A_1j = H_0 + H_1j$ then \[
AC = (H_0 + H_1j)(H_0U_0 + H_1U_1j) = (H_0 + H_1j)(H_0U_0 + H_1U_1j) = (H_UU_0 + H_UU_1j)(H_U^T + H_U^Tj)
\]
\[
= U_0(H_U^T)^2 + U_1(H_U^T)^2j
\]
Therefore $AC$ is quaternion quasinormal.

**Theorem: 3**

If $A = G_0W_0 + G_Wj = W_0G_0 + W_Gj$ is quaternion normal and $B = S_0V_0 + S_1V_1j = V_0S_0^T + V_1S_1^Tj$ is quaternion quasi-normal (where $G_0, G_1, S_0, S_1$ are hermitian and $W_0, W_1, V_0, V_1$ are unitary) then $AB$ is quaternion quasi-normal if and only if $G_0S_0 + G_1S_1j = S_0G_0 + S_1G_1j, G_0V_0 + G_1V_1j = V_0G_0^T + V_1G_1^Tj$ and $W_0S_0 + W_1S_1j = S_0W_0 + S_1W_1j$.

**Proof:**

If the three relations hold, then \[
AB = G_0W_0S_0V_0 + G_1W_1S_1V_1j = G_0S_0W_0V_0 + G_1S_1W_1V_1j
\]
on one hand, and $AB$ is quaternion quasi-normal, since $G_0S_0 + G_1S_1j$ is hermitian and $W_0V_0 + W_1V_1j$ is unitary.

Conversely, let $A = U_0^CD_0U_0 - U_1^CD_1U_1j = G_0W_0 + G_1W_1j$ and $B = U_0^CB_0^T + U_1^CB_1^TU_1j$ \[
=(U_0^CS_0U_0 - U_1^CS_1U_1j)(U_0^CV_0U_0 + U_1^CV_1U_1j) = V_0S_0^T + V_1S_1^Tj
\]
where $S_0(0), S_1(1)$ and $V_0(0), V_1(1)$ are hermitian and unitary and direct sums conformable to $B_0^T + B_1^TU_1j$.

A direct check shows that $G_0S_0 + G_1S_1j = S_0G_0 + S_1G_1j$ and $G_0V_0 + G_1V_1j = V_0G_0^T + V_1G_1^Tj$ also $W_0S_0 + W_1S_1j$ \[
= U_0^CD_0U_0 - U_1^CD_1U_1j = U_0^CK_0(0)U_0 - U_1^CK_1(1)U_1j = S_0W_0 + S_1W_1j
\]
Since
If \( A_0 + A_i j \) is quaternion normal, \( B_0 + B_i j \) is quaternion quasi-normal, and \( A_0B_0 + A_iB_i j = B_0A_0^T + B_iA_i^T j \), then

\[
W_0A_0W_0^T - W_0A_1W_1^T j = D_0 + D_1 j \quad \text{and} \quad W_0B_0W_0^T + W_1B_1W_1^T j = F_0 + F_1 j ,
\]

the quaternion normal form of Theorem 1, where \( W_0 + W_1 j \) is a quaternion unitary matrix; also \( A_0B_0 + A_iB_i j \) is quaternion quasi-normal.

**Proof:**

Let \( U_0A_0U_0^T - U_1A_1U_1^T j = D_0 + D_1 j \) quaternion diagonal and \( U_0B_0U_0^T + U_1B_1U_1^T j = B_0(2) + B_1(2) j \) which is quaternion quasi-normal. Then \( A_0B_0 + A_iB_i j = B_0A_0^T + B_iA_i^T j \), implies \( D_0B_0(2) + D_1B_1(2) j \).

\[
= U_0A_0U_0^T U_0B_0U_0^T - U_1A_1U_1^T U_1B_1U_1^T j \quad \text{is quaternion normal.}
\]

Let \( D_0 + D_1 j = C_1I_1 \oplus C_2I_2 \oplus \cdots \oplus C_mI_m \), where the \( C_p \) are quaternion and \( C_p \neq C_q \) for \( p \neq q \) and \( C_{(p)} \), \( C_{(q)} \), \( B_0(2) + B_1(2) j = C_1 \oplus C_2 \oplus \cdots \oplus C_m \). Let \( V_p \) be unitary such that \( V_0C_0V_0^T + V_1C_1V_1^T j = F_0(2) + F_1(2) j = \) the real quaternion normal form of Theorem 1, and let \( V_0 + V_1 j = V_1 \oplus V_2 \oplus \cdots \oplus V_m \). Then

\[
V_0U_0U_0^T V_0^T + V_1U_1U_1^T V_1^T j = D_0 + D_1 j . \quad \text{And} \quad V_0U_0U_0^T V_1^T + V_1U_1U_1^T V_0^T j = F_0 + F_1 j .
\]

Also \( A_0B_0 + A_iB_i j = B_0A_0^T + B_iA_i^T j \) implies that \( B_0^T A_0^T + B_i^T A_i^T j = A_0B_0^T + A_iB_i^T j \) and so

\[
A_0B_0^T A_0^T + A_iB_i^T A_i^T j = B_0A_0^T A_0^T + B_iA_i^T A_i^T j = B_0C_0C_0^T + B_iA_i^T A_i^T j .
\]

It is also possible for the product of two quaternion normal matrices \( A_0 + A_i j \) and \( B_0 + B_i j \) to be quaternion normal. Let \( Q_0 + Q_1 j = H_0U_0 + H_1U_1 j = U_0H_0^T + U_1H_1^T j \) is quaternion quasi-normal and if \( A_0 + A_i j = U_0 + U_1 j \) and

\[
B_0 + B_i j = H_0 + H_1 j \quad \text{this is so or if} \quad S_0V_0 + S_1V_1 j = V_0S_0^T + V_1S_1^T j \quad \text{is quaternion quasi-normal and if} \quad A_0 + A_i j = U_0S_0 + U_1S_1 j \quad \text{is quaternion normal with} \quad S_0 \quad \text{and} \quad S_1 \quad \text{are hermitian and}
\]

\[
V_0 + V_1 j \quad \text{is quaternion unitary for} \quad B_0 + B_i j \quad \text{we have} \quad A_0B_0 + A_iB_i j = (U_0S_0 + U_1S_1 j)(V_0 + V_1 j) = (S_0 + S_1 j)(U_0V_0 + U_1V_1 j) = U_0V_0S_0^T + U_1V_1S_1^T j \quad \text{is a quaternion quasi-normal.}
\]

But if in the first example \( U_0^2H_0 + U_1^2H_1 j \) is not quaternion normal, then \( H_0U_0 + H_1U_1 j \) is not quaternion normal. So that \( B_0A_0 + B_1A_i j \) is not necessarily quaternion quasi-normal though \( A_0B_0 + A_iB_i j \) is. When \( A_0 + A_i j \) alone is quaternion normal an analog of Theorem 2 can be obtained which states the following: If \( A_0 + A_i j \) is quaternion normal then

\[
A_0B_0 + A_iB_i j \quad \text{and} \quad A_0B_0^T + A_iB_i^T j \quad \text{are quaternion quasi-normal if and only if} \quad A_0B_0^T + A_iB_i^T j = B_0^T A_0B_0 + B_i^T A_iB_i j = A_0B_0^T + A_iB_i^T j \].
\]

(The proof is not included here because of its similarity to that above).
It is possible for the product of two quaternion quasi-normal matrices to be quaternion quasi-normal, but no such simple analogous necessary and sufficient conditions as exhibited above are available. This may be seen as follows. Two non-real quaternion commutative matrices \( X_0 + X_1 j \) and \( Y_0 + Y_1 j \) can form a quaternion quasi-normal (and non-real symmetric) matrix \( X_0 Y_0 + X_1 Y_1 j \) (such that \( Y_0 X_0 + Y_1 X_1 j \) is also quaternion quasi-normal) which need not be quaternion normal.

Then two symmetric matrices:

\[
X_0 + X_1 j = \begin{bmatrix}
i & 1+i \\
1+i & -i
\end{bmatrix},
Y_0 + Y_1 j = \begin{bmatrix}
1+2i & 3-4i \\
3-4i & -(1+2i)
\end{bmatrix}
\]

Are such that \( X_0 Y_0 + X_1 Y_1 j = Z_0 + Z_1 j \) is real, quaternion normal and quaternion quasi-normal (and not symmetric). Finally, if \( U_0 + U_1 j \) and \( V_0 + V_1 j \) are two quaternion unitary matrices of the same order, they can be chosen so \( U_0 V_0 + U_1 V_1 j \) is non-real quaternion, quaternion normal and quaternion quasi-normal.

If \( A_0 + A_1 j = (X_0 + X_1 j) + (S_0 + S_1 j) + (U_0 + U_1 j) \), \( B_0 + B_1 j = (Y_0 + Y_1 j) + (T_0 + T_1 j) + (V_0 + V_1 j) \). Then \( A_0 B_0 + A_1 B_1 j = (X_0 Y_0 + X_1 Y_1 j) + (S_0 T_0 + S_1 T_1 j) + (U_0 V_0 + U_1 V_1 j) \) where \( A_0 + A_1 j \) and \( B_0 + B_1 j \) are quaternion quasi-normal as in \( A_0 B_0 + A_1 B_1 j \) (but not symmetric).

A simple inspection of these matrices shows that relations on the order of \( (B_0^T B_0^C)A_0 - (B_0^T B_1^C)A_1 j = (A_0^C B_0^T B_0^C - A_0^C B_1^T B_0^C)j \) do not necessarily hold; these are sufficient, however, to guarantee that \( A_0 B_0 + A_1 B_1 j \) is quaternion quasi-normal (as direct verification from the definition will show).

References:


