ON THE STABILITY OF SECOND ORDER LINEAR DIFFERENCE AND LINEAR FUNCTIONAL EQUATIONS

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ABSTRACT:

In this work, on the stability of second order linear difference and linear functional equations of the form:

\[ x_{n+2} + \gamma x_{n+1} + \delta x_n = 0, \]
\[ x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n \]

and

\[ x_{n+2} - \gamma_n x_{n+1} + \delta_n x_n = p_n \]

are studied, where \( \gamma, \delta \in \mathbb{R} \) and \( p_n, \gamma_n, \delta_n \) are sequence of reals.

1. INTRODUCTION

On the stability problem for functional equations was replaced by stability of differential equations. The differential equation

\[ r_n(t)x^{(n)}(t) + r_{n-1}(t)x^{(n-1)}(t) + \cdots + r_1(t)x'(t) + r_0(t)x(t) + h(t) = 0 \]

has the stability, if for given \( \varepsilon > 0 \), I be an interval and for any function \( g \) satisfying the differential inequality

\[ |r_n(t)x^{(n)}(t) + r_{n-1}(t)x^{(n-1)}(t) + \cdots + r_1(t)x'(t) + r_0(t)x(t) + h(t)| \leq \varepsilon \]

then there exists a solution \( g_0(t) \) of the above equation such that

\[ |g(t) - g_0(t)| \leq L(\varepsilon) \]

\[ \lim_{\varepsilon \to 0} L(\varepsilon) = 0, t \in I \]

We have discussed on the stability of second order linear differential and linear functional equations of the form:

\[ x'' + ax' + bx = 0 \quad \text{(1.1)} \]

and

\[ x'' + ax' + bx = g(t) \quad \text{(1.2)} \]

Where \( a, b \in \mathbb{R} \). The objective of this work is to study the stability of discrete analogue of the equations (1.1) and (1.2) as

\[ x_{n+2} + \gamma x_{n+1} + \delta x_n = 0 \quad \text{(1.3)} \]

and

\[ x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n \quad \text{(1.4)} \]
Where \( \gamma, \delta \in \mathbb{R} \) and \( p_n \) is a sequence of reals. Also, an effort is made to study on the stability of \( x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n \) \((1.5)\)

**DEFINITION: 1.1**

The difference equation
\[
r_k(n)x(n+k) + r_{k-1}(n)x(n+k-1) + \cdots + r_1(n)x(n+1) + r_0(n)x(n) + h(n) = 0
\]
has the stability, if for given \( \epsilon > 0 \), \( I \) be an open interval and for any function \( g \) satisfying the inequality
\[
|g(n) - g_0(n)| \leq L(\epsilon) \text{ for } n \in I \subset N(0) = \{0, 1, 2, 3, \ldots \}.
\]

**DEFINITION: 1.2**

We say that (1.4) has the stability if there exists a constant \( L > 0 \) with the property: for every \( \epsilon > 0 \), \( x_n, p_n \) defined for \( n \in (r, s + 1) \), \( 0 < r < s < \infty \), if
\[
|x_{n+2} + \gamma x_{n+1} + \delta x_n - p_n| \leq \epsilon, \quad \text{-------- (1.6)}
\]
Then there exists some \( z_n, n \in (r, s + 1) \) satisfying
\[
z_{n+2} + \gamma z_{n+1} + \delta z_n = p_n
\]
Such that \( |x_n - z_n| < L \epsilon \). Let \( L \) be a Hyers–Ulam stability constant for (1.4).

**2. STABILITY RESULTS FOR** \( x_{n+2} + \gamma x_{n+1} + \delta x_n = 0 \) **and** \( x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n \)

Now, in this section deals with the stability of \( x_{n+2} + \gamma x_{n+1} + \delta x_n = 0 \) and \( x_{n+2} + \gamma x_{n+1} + \delta x_n = p_n \).

**THEOREM: 2.1**

Assume that the characteristic equation \( m^2 + \gamma m + \delta = 0 \) have two different positive roots. Then (1.3) has the stability.

**Proof:**

Let \( \epsilon > 0 \) and \( x_n, n \in (r, s + 1) \) be a solution of (1.3) satisfying the property
\[
|x_{n+2} + \gamma x_{n+1} + \delta x_n| \leq \epsilon.
\]

Let \( \lambda \) and \( \mu \) be the positive roots of the characteristic equation. For \( n \in (r, s + 1) \), define \( f_n = x_{n+1} - \lambda x_n \). Then
\[
f_{n+1} = x_{n+2} - \lambda x_{n+1}
\]
and hence
\[
|f_{n+1} - f_n| = |x_{n+2} - \lambda x_{n+1} - \mu x_{n+1} + \lambda \mu x_n|
\]
\[
= |x_{n+2} - (\lambda + \mu) x_{n+1} + \lambda \mu x_n|
\]
\[
= |x_{n+2} + \gamma x_{n+1} + \delta x_n| \leq \epsilon.
\]

Equivalently, \( f_n \) satisfies the relation
\[
-\epsilon \leq f_{n+1} - f_n \leq \epsilon \quad \text{-------- (2.1)}.
\]

Upon the choice of \( \lambda \) and \( \mu \), We have four possibilities.

\( i) \lambda > 1, \mu > 1; \quad ii) \lambda \leq 1, \mu > 1; \quad iii) \lambda > 1, \mu \leq 1; \quad iv) \lambda \leq 1, \mu > 1\)

Consider case \( i) \)

Then (2.1) can be viewed as
\[
-\epsilon \mu^{-(n+1)} \leq \mu^{-(n+1)}|f_{n+1} - \mu f_n| \leq \epsilon \mu^{-(n+1)},
\]

i.e.
\[
-\epsilon \mu^{-(n+1)} \leq \Delta(\mu^{-n}f_n) \leq \epsilon \mu^{-(n+1)} \quad \text{-------- (2.2)}
\]

Therefore for \( n \in (r, s + 1) \), it follows that
\[-\varepsilon \sum_{j=n}^{s} \mu^{-(j+1)} \leq \sum_{j=n}^{s} \Delta(f_{j}) \leq \varepsilon \sum_{j=n}^{s} \mu^{-(j+1)} \]

Which is implies that
\[-\varepsilon \mu^{-n} \leq \mu^{-(s+1)} f_{s+1} - \mu^{-n} f_{n} \leq \varepsilon \mu^{-n}.\]

Consequently,
\[-\varepsilon_{1} \leq \mu^{-(s-n+1)} f_{s+n} - f_{n} \leq \varepsilon_{1},\]

Where \( \varepsilon_{1} = \frac{\varepsilon}{\mu-1} \).

Let \( z_{n} = \mu^{-(s-n+1)} f_{s+n} \).

Then \( z_{n+1} - \mu z_{n} = 0 \). Now, \( |f_{n} - z_{n}| \leq \varepsilon_{1} \) implies that
\[-\varepsilon_{1} \leq x_{n+1} - \lambda x_{n} - z_{n} \leq \varepsilon_{1},\]

and hence
\[-\varepsilon_{1} \lambda^{-(n+1)} \leq \lambda^{-(n+1)}(x_{n+1} - \lambda x_{n} - z_{n}) \leq \varepsilon_{1} \lambda^{-(n+1)} \lambda^{-1}.\]

Proceeding as above, we obtain
\[-\varepsilon_{1} \lambda^{-n} \leq \lambda^{-(s+1)} x_{s+1} - \lambda^{-n} x_{n} - \sum_{j=n}^{s} \lambda^{-(j+1)} z_{j} \leq \varepsilon_{1} \lambda^{-n} \lambda^{-1}.\]

(i.e.,
\[-\varepsilon_{1} \lambda^{-n} \leq \lambda^{-(s-n+1)} x_{s+n+1} - x_{n} - \lambda^{n} \sum_{j=n}^{s} \lambda^{-(j+1)} z_{j} \leq \varepsilon_{1} \lambda^{-n} \lambda^{-1}.\]

Denote that,
\[ u_{n} = \lambda^{-(s-n+1)} x_{s+n+1} - \sum_{j=n}^{s} \lambda^{-(j-n+1)} z_{j}.\]

Then \( |u_{n} - x_{n}| \leq \frac{\varepsilon}{\lambda-1} = \frac{\varepsilon}{(\lambda-1)(\mu-1)}.\)

It is easy to verify that \( u_{n+1} = \lambda u_{n} + z_{n} \) and hence
\[ u_{n+2} - \lambda u_{n+1} = z_{n+1} = \mu z_{n} = \mu[u_{n+1} - \lambda u_{n}] \]

implies that
\[ u_{n+2} + \gamma u_{n+1} + \delta u_{n} = 0.\]

Consequently, (1.3) has the stability with the stability constant
\[ L = \frac{1}{(\lambda-1)(\mu-1)} \]

Next, we consider Case (ii).

Assume that there exist positive integers \( M,N > 0 \) such that \( \mu M > 1 \) and \( \lambda N > 1 \).

Using the same type of argument as in case (i), we get the equation (2.2) and hence
\[ -\varepsilon \sum_{j=n}^{s} (\mu M)^{-(j+1)} M^{j+1} \leq \sum_{j=n}^{s} \Delta(\mu^{-j} f_{j}) \leq \varepsilon \sum_{j=n}^{s} (\mu M)^{-(j+1)} M^{j+1} \]

Then (2.3) becomes
\[-\varepsilon M^{*} \sum_{j=n}^{s} (\mu M)^{-(j+1)} \leq \mu^{-(s+1)} f_{s+1} - \mu^{-n} f_{n} \leq \varepsilon M^{*} \sum_{j=n}^{s} (\mu M)^{-(j+1)}, \]

(i.e.,
\[ M^{*} - \varepsilon (\mu M)^{-n} \leq \mu^{-(s+1)} f_{s+1} - \mu^{-n} f_{n} \leq M^{*} \varepsilon (\mu M)^{-n}. \]
Consequently,

\[
\frac{-\epsilon M^*}{(\mu M - 1)M^*} \leq \mu^{-(s-n+1)} f_{n+1} - f_n
\]

\[
\leq \frac{-\epsilon M^*}{(\mu M - 1)M^*}
\]

The rest of the proof follows from Case (i). We note that the stability constant is given by

\[
K = \frac{e^{\mu N^*}}{(\mu M - 1)(M - 1)^{MN + 1}}
\]

Where

\[
N^* = \max\{N^{n+1}, N^{n+2}, \ldots, N^{s+1}\}, r \leq n < s + 1.\]

Cases (iii) and (iv) follow from Cases (i) and (ii).

**Hence the proof.**

**THEOREM 2.2:**

Assume that the characteristic equation

\[
m^2 + \gamma m + \delta = 0
\]

have two different positive roots. Furthermore, assume that (1.6) holds. Then (1.4) has the Hyers-Ulam stability.

**Proof:**

Proceeding as in the proof of theorem 2.1, we obtain

\[
|f_{n+1} - \mu f_n - p_n| = |x_{n+2} - \lambda x_{n+1} - \mu x_n + \lambda \mu x_n - p_n|
\]

\[
= |x_{n+2} - (\lambda + \mu) x_{n+1} + \lambda \mu x_n - p_n|
\]

\[
= |x_{n+2} + \gamma x_{n+1} + \delta x_n - p_n| \leq \epsilon.
\]

Equivalently, \(f_n\) satisfies the relation

\[-\epsilon \leq f_{n+1} - \mu f_n - p_n \leq \epsilon.
\]

Similar to Theorem 2.1, we have four possibilities upon the choices of \(\lambda\) and \(\mu\).

We consider Case (i) only.

And hence, similar to the equation (2.2),

\[
\text{We have}
\]

\[
-\epsilon \mu^{-(n+1)} \leq \Delta(\mu^{-n} f_n) - \mu^{-(n+1)} p_n
\]

\[
\leq \epsilon \mu^{-(n+1)}
\]

and let

\[
z_n = \mu^{-(s-n+1)} f_{s+1} - \mu^n \sum_{j=n}^{s} \mu^{-(j+1)} p_j
\]

Therefore, \(z_n\) satisfies \(z_{n+1} - \mu z_n - p_n = 0\), and \(|f_n - z_n| \leq \epsilon_1\).

Using the same type of argument as in Theorem 2.1, we can show that there exists

\[
u_n = \lambda^{-(s-n+1)} x_{n+1} - \sum_{j=n}^{s} \lambda^{-(j-n+1)} z_j
\]

such that \(|u_n - x_n| \leq \epsilon\) and \(u_n\) satisfies

\[
u_{n+2} + \gamma u_{n+1} + \delta u_n - p_n = 0.
\]

**Hence the proof.**

**REFERENCE**


