ON CHARACTERIZATION OF \( k \cdot J \cdot EP \) MATRICES

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Abstract

The Concept of \( k \cdot J \cdot EP \) matrices is introduced. Relation between \( k \cdot EP \) and \( EP \) matrices are discussed. As a generalization of \( k \)-Hermitian, \( k \cdot EP \) matrices and \( J \cdot EP \) matrices. Necessary and Sufficient Condition are determined for a matrix to be \( k \cdot J \cdot EP \) (\( k \cdot EP \), \( J \cdot EP \) and rank \( r \)). Equivalent Characterization of \( k \cdot J \cdot EP \) matrix are discussed. As an application, we shown that the class of all \( J \cdot EP \) and \( k \cdot EP \) having the same range space form a group under multiplication.

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Introduction

In matrix theory, we come across many special types of matrices and one among them is the normal matrix, which plays an important role in the spectral theory of rectangular matrices. In 1918, the concept of a normal matrix with entries from the complex field was introduced by O.Toeplitz [11] who gave a necessary and sufficient condition for a complex matrix to be normal [1,3,4]. As a generalization of normality the concept of \( EP \) matrices over the complex field was introduced by Schwerdtfeger [10]. The class of complex \( EP \) matrices includes the class of all non singular matrices, Hermitian matrices and normal matrices. The concept of \( k \cdot EP \) [7] matrix is introduced for complex matrices and exhibited as a generalization of \( k \)-Hermitian and \( EP \) matrices.

An indefinite inner product in \( C^n \) is a conjugate symmetric sesquilinear form \([x, y] \) together with the regularity condition that \([x, y] = 0, \forall y \in C^n \) only when \( x = 0 \). Associated with any indefinite inner product is a unique invertible Hermitian matrix \( J \) (called a weight with Complex entries such that \([x, y] = \langle x, Jy \rangle \). Where \( \langle x, y \rangle \) denotes the Euclidean inner product of \( C^n \) and vice versa.

Motivated by the notion of Minkowski space (as studied by physicists), we also make an additional assumption on namely \( J^2 = I \) [9]. Investigation of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors (see for instance [2,8]).

More precisely, the indefinite matrix product of two matrices \( A \) and \( B \) of sizes \( m \times n \) and \( n \times l \) complex matrices respectively, is denoted to be the matrix \( AB = AJ_n J_m B \). The adjoint of \( A \), denoted by \( A^* \) is defined to be the matrix \( J_m A^* J_n \) where \( J_m \) and \( J_n \) weights in appropriate spaces.

Definition 1.1:

\[ A \in C_{evn} \text{ is said to be } k \cdot J \cdot EP \text{ if it satisfies the condition equivalently } N(A) = N(KA^* K) = N(KJA^* JK) \. \text{ A is said to be } k \cdot J \cdot EP_r \text{, if } A \text{ is } k \cdot J \cdot EP \text{ and of rank } r \text{ where } A^* = JA^* J \.\]

In particular, when \( k(i) = i \) and \( j(i) = i \), for each \( i = 1 \) to \( n \), then the associated permutation matrix \( K \) reduces to the identity matrix and definition (1.1) reduces to \( N(A) = N(A^*) \) which implies that \( A \) is \( EP \) matrix [10]. If \( A \) is non singular, then \( A \) is \( k \cdot J \cdot EP \) for all transpositions \( k \) in \( S_n \).
Remark 1.2:

We note that a \( k - J \)-Hermitian matrix \( A \) is \( k - J - EP \). For, if \( A \) is \( k - J \)-Hermitian, then by known theorem, \( A = K A^{[k]} K = K J A^{\ast} J K \). Hence \( N(A) = N(K A^{[k]} K) = N(A^{\ast} J K) \) which implies \( A \) is \( k - J - EP \). However the converse need not be true.

Example 1.3:

Let

\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

For a transposition \( k = (1 2 3) \) the associated permutation matrix \( K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). \( K J A^{\ast} J K \neq A \). Therefore \( A \) is not \( k - J \)-Hermitian. \( A \) being a non-singular matrix, is a \( k - J - EP \) matrix. Thus, the class of \( k - J - EP \) matrices is a wider class containing the class of \( k - J \)-Hermitian matrices.

Theorem 1.4:

For \( A \in C_{nn} \) the following are equivalent:

1. \( A \) is \( k - J - EP \)
2. \( K J A \) is \( EP \)
3. \( A J K \) is \( EP \)
4. \( A^{\ast} \) is \( k - J - EP \)
5. \( N(A) = N(A^{\ast} J K) \)
6. \( N(A^{\ast}) = N(A J K) \)
7. \( R(A) = R(K J A^{\ast}) \)
8. \( R(A^{\ast}) = R(K J A) \)
9. \( K J A^{\ast} A = A A^{\ast} J K \)
10. \( A^{\ast} J K A = K J A A^{\ast} \)
11. \( A = K J A^{\ast} J K H \) for a nonsingular \( n \times n \) matrix \( H \).
12. \( A = H K J A^{\ast} J K \) for a nonsingular \( n \times n \) matrix \( H \).
13. \( A^{\ast} = H K J A^{\ast} J K \) for a nonsingular \( n \times n \) matrix \( H \).
14. \( A^{\ast} = K J A J K H \) for a nonsingular \( n \times n \) matrix \( H \).
15. \( C_{n} = R(A) \oplus N(A J K) \)
16. \( C_{n} = R(K J A) \oplus N(A) \)
Proof:

The proof for the equivalence of (1), (2) and (3) runs as follows:

\[ A \text{ is } k \cdot J \cdot EP \iff N(A) = N(A^\dagger JK) \iff N(A) = N(KJA)^\dagger \iff KJA \text{ is } EP \iff (KJ)(KJA)(KJ) \text{ is } EP \iff AJK \text{ is } EP. \]

Thus (1) \iff (2) \iff (3) hold.

(2) \iff (4)

\[ KJA \text{ is } EP \iff (KJA)^\dagger \text{ is } EP \iff A^\dagger JK \text{ is } EP \iff A^\dagger \text{ is } k \cdot J \cdot EP \]

By equivalence of (1) and (3) applied to \( A \).

Thus equivalence of (2) and (4) is proved.

(1) \iff (5)

\[ A \text{ is } k \cdot J \cdot EP \iff N(A) = N(A^\dagger JK) \iff N(A) = N(KJA)^\dagger \iff N(A) = N(KJA)^\dagger \iff N(A) = N(A^\dagger JK). \]

Thus, equivalence of (1) and (5) is proved. The other equivalences with (2) can be proved along the same lines and hence the proof is omitted.

Remark 1.5:

In particular, when \( A \) is \( k \cdot J \)-Hermitian, then theorem (1.4) reduces to result 2.1 of [3]. In result 2.8 of [3]. It is stated that \( A^\dagger \) is \( k \cdot J \)-Hermitian for a \( k \cdot J \)-Hermitian matrix \( A \) only when \( A \) is normal. We note that, without any condition on \( A \) to be normal the result follows:

\[ A \text{ is } k \cdot J \text{-Hermitian.} \iff A = KJA^\dagger JK \iff A^\dagger = (KJA^\dagger JK)^\dagger \iff A^\dagger = KJ(A^\dagger)^\dagger JK = KJ(A^\dagger)^\dagger JK \iff A^\dagger \text{ is } k \cdot J \cdot \text{Hermitian.} \]

When \( k(i) = i \), \( j(i) = i \), for each \( i = 1 \) to \( n \), then theorem (1.4) reduces to theorem 1 of [4]. Further, the class of complex normal matrices is a subclass of \( EP \) matrices. However this is not the case with \( k \cdot J \cdot EP \) matrices.

Example 1.6:

Let \( K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) and \( A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

(i) \( A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) is \( EP \) as well as \( k \cdot J \cdot EP \).

Let \( K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) and \( A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

(ii) \( A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) is \( k \cdot J \cdot EP \) but not \( EP \).

Let \( K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) and \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \)
Theorem 1.7:

Let \( A \in C_{nn} \), then any two of the following conditions imply the other one:

1. \( A \) is \( EP \).
2. \( A \) is \( k \cdot J \cdot EP \).
3. \( R(A) = R(KJA) \).

Proof:

First, we prove that, whenever (1) holds, then (2) and (3) are equivalent. Suppose (1) holds; then by definition of \( EP \) matrix, \( R(A) = R(A') \), now by theorem (1.4), \( A \) is \( k \cdot J \cdot EP \iff R(A') = R(KJA) \). Therefore, \( A \) is \( k \cdot J \cdot EP \iff R(A) = R(KJA) \). This completes the proof of (1) and (2) \( \Rightarrow \) (3); (1) and (3) \( \Rightarrow \) (2).

Remark 1.8:

For \( O \neq E \neq I_n \), \( H_n(E) \) is a non abelian group \( \iff n > 2 \). For \( A \in C_{nn} \), there exist unique \( k \cdot J \cdot \) Hermitian matrices \( P \) and \( Q \) such that \( A = P + iQ \), where \( P = (1/2)(A + KJA^*JK) \) and \( Q = (1/2)(A - KJA^*JK) \). In the following theorem, an equivalent condition for a matrix to be \( k \cdot J \cdot EP \) is obtained in terms of \( P \), the \( k \cdot J \cdot \) Hermitian part of \( A \).

Theorem 1.9:

For \( A \in C_{nn} \), \( A \) is \( k \cdot J \cdot EP \iff N(A) \subseteq N(P) \). Where \( P \) is the \( k \cdot J \cdot \) Hermitian part of \( A \).

Proof:

If \( A \) is \( k \cdot J \cdot EP \), then by theorem (1.4), \( KJA \) is \( EP \). Since \( K \) is non singular, we have \( N(A) = N(KJA) = N((KJA)^* = N(A^*JK) = N(KJA^*JK) \). Then, for \( x \in N(A) \), both \( KJAx = 0 \) and \( KJA^*JKx = 0 \) which implies that \( Px = (1/2)(A + KJA^*JK)x = 0 \). Thus \( N(A) \subseteq N(P) \).

Conversely, let \( N(A) \subseteq N(P) \), then \( Ax = 0 \) implies \( Px = 0 \) hence \( Qx = 0 \). Therefore, \( N(A) \subseteq N(Q) \). Thus, \( N(A) \subseteq N(P) \cap N(Q) \). Since both \( P \) and \( Q \) are \( k \cdot J \cdot \) Hermitian, by \( P = KJP^*JK \) and \( Q = KJQ^*JK \). Hence \( N(P) = N(KP^*JK) = N(P^*JK) \) and \( N(Q) = N(KP^*JK) = N(P^*JK) \). Now \( N(A) \subseteq N(P) \cap N(Q) = N(P^*JK) \cap N(Q^*JK) \subseteq N(P^* - iQ^*)JK \). Therefore \( N(A) \subseteq N(A^*JK) \) and \( rk(A) = rk(A^*JK) \). Hence \( N(A) \subseteq N(A^*JK) \). Therefore \( A \) is \( k \cdot J \cdot EP \). Hence the theorem.

Next, we derive equivalent conditions for a matrix to be \( k \cdot J \cdot EP \). To make the proof simpler, first let us prove certain lemmas.

Lemma 1.10:

Let \( B \in C_{nn} \) be of the form \( B = \begin{pmatrix} D & O \\ O & O \end{pmatrix} \) where \( D \) is a non singular matrix. Then the following are equivalent:

1. \( B \) is \( k \cdot J \cdot EP \).
2. \( R(KJB) = R(B) \).
(3) $BB^* \equiv k \cdot J \cdot EP$

(4) $KJ = \begin{pmatrix} K_1J_1 & O \\ O & K_2J_2 \end{pmatrix}$ where $K_1$ and $K_2$ are permutation matrices of order $r$ and $(n-r)$ respectively, and $J_1$ and $J_2$ are of order $r$ and $(n-r)$ respectively with $J_i = J_i^* = J_i^{-1}$, $i = 1$ to $2$.

(5) $kJ = k_1J_1k_2J_2$ where $k_1$ is the product of disjoint transpositions in $S_n$ leaving $(r+1, r+2, \ldots, n)$ fixed and $k_2$ is the product of disjoint transpositions leaving $(1,2,\ldots,r)$ fixed and $J_1$ and $J_2$ of orders $r$ and $(n-r)$ respectively with $J_i = J_i^* = J_i^{-1}$, $i = 1$ to $2$.

**Proof:**

(1) $\iff$ (2) Since $B$ is $J - EP$, $B$ is $J - EP$ with $rk(B) = r \iff R(B) = R(JB^*)$. Therefore $B$ is $k \cdot J \cdot EP \iff R(B') = R(KJB)$. Therefore $B$ is $k \cdot J \cdot EP \iff R(B') = R(KJB)$

(2) $\iff$ (3) Since $R(KJB) = R(B)$, $R(A) = R(KJA^*)$ and $rk(B) = r$ (since $A$ is $k \cdot J \cdot EP$). Therefore $BB^*$ is $k \cdot J \cdot EP$.

(2) $\iff$ (4) $R(KJB) = R(B) \iff (KJB)(KJB)^\dagger = BB^\dagger \iff KJ(BB^\dagger)JK = BB\dagger$ $\iff KJBK$.

Let us partition $K = \begin{pmatrix} K_1 & K_\perp \\ K_\perp^T & K_2 \end{pmatrix}$ where $K_1$ is $r \times r$, and $J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$ where $J_1$ is $r \times r$.

$KJBK = BB^\dagger JK \iff KJ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ $\iff \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ $\iff \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ $\iff \begin{pmatrix} K_1J_1 \\ K_3J_3 \end{pmatrix}$ $\iff \begin{pmatrix} K_1J_1 \\ K_3J_3 \end{pmatrix}$ $\iff \begin{pmatrix} K_1J_1 \\ K_3J_3 \end{pmatrix}$ $\iff KJ$.

Thus, equivalence of (2) and (4) holds. The equivalence of (4) and (5) is clear from the definition of $'k \cdot J'$. 

**Remark 1.11:**

In Lemma (1.4), the form of $B$ is essential can be seen by the following example:

$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is not $EP$. For $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $KJB$ is $EP$. 


Lemma 1.12:

\[ A \in C_{n \times n} \text{ is } k \cdot J \cdot EP_r \iff \text{ there exist unitary matrix } U \text{ and } r \times r \text{ non singular matrix } F \text{ such that} \]

\[ A = KJU \begin{pmatrix} F & O \\ O & O \end{pmatrix} U^*. \]

Proof:

Let us assume that \( A \) is \( k \cdot J \cdot EP_r \). Then by theorem (1.4), \( A' = HKJAJK \), where \( H \) is a non singular matrix. Now, \( A' A = (HKJAJK)A = H(KJA)(KJA) = H(KJA)^2 \Rightarrow rk(H(KJA)^2) \Rightarrow rk(A' A) = rk(KJA)^2 \). Over the complex field and \( A \) have the same rank. Therefore \( rk(KJA)^2 = rk(A' A) = rk(A) = rk(KJA) \Rightarrow R(KJA) \cap N(KJA) = \{0\} \Rightarrow R(KJA) \cap N(A) = \{0\} \). Thus \( C_n = R(KJA) \oplus N(A) \).

Choose an orthonormal basis \( \{x_1, x_2, \ldots, x_n\} \) of \( R(KJA) = R(A') \) extend it to be a basis \( \{x_1, x_2, x_3, \ldots, x_r, y_1, \ldots, y_{n-r}\} \) of \( C_n \), where \( \{x_1, x_2, \ldots, x_r\} \) orthonormal basis of \( N(A) \). If \((u,v)\) denotes the usual inner product of \( C_n \) and \( 1 \leq r < j \leq n \), it follows that \( x_j \in R(KJA) = R(A') \Rightarrow x_j = A'y \). Therefore \((x_i, x_j) = (A'y, x_j) = (y, Ax_j) = 0 \) (since \( x_j \in N(A) \)). Hence \( \{x_1, x_2, x_3, \ldots, x_r\} \) is an orthonormal basis of \( C_n \). If we consider \( KJA \) as the matrix of a linear transposition relative to any orthonormal basis of \( C_n \) then \( U^{*}(KJA)U = \begin{pmatrix} F & O \\ O & O \end{pmatrix} \) Where \( F \) is \( r \times r \) non singular matrix. Implies that \( A = KJU \begin{pmatrix} F & O \\ O & O \end{pmatrix} U^* \).

Conversely, if \( A = KJU \begin{pmatrix} F & O \\ O & O \end{pmatrix} \), by theorem (1.4), \( A \) is \( k \cdot J \cdot EP_r \).

Theorem 1.13:

\[ A \in C_{n \times n} \text{ is } k \cdot J \cdot EP_r \text{ where } k = k_1k_2 \text{ and } J = \begin{pmatrix} J_1 & O \\ O & J_2 \end{pmatrix} \text{ order of } J_1 \text{ is } r \times r, \text{ order of } J_2 \text{ is } (n-r) \text{ iff } A \text{ is unitarily } k \cdot J \text{-similar to a diagonal block } k \cdot J \cdot EP_r \text{ matrix } B = \begin{pmatrix} D & O \\ O & O \end{pmatrix} \text{ where } D \text{ is } r \times r \text{ non singular matrix. (det } D = 0). \]

Proof:

Since \( A \) is \( k \cdot J \cdot EP_r \), by lemma (1.12), there exist unitary matrix \( U \) and a \( r \times r \) non singular matrix \( F \) such that \( A = (KUJK)(KJ) \begin{pmatrix} F & O \\ O & O \end{pmatrix} U^* \). Since \( k = k_1k_2 \), the associated permutation matrix is \( K = \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix} \) and \( J = \begin{pmatrix} J_1 & O \\ O & J_2 \end{pmatrix} \). Hence \( A = (KUJK) \begin{pmatrix} K_1J_1F & O \\ O & O \end{pmatrix} U^* = KJUJK \begin{pmatrix} D & O \\ O & O \end{pmatrix} U^* \), where \( D = K_1J_1F \). Thus \( A \) is unitarily \( k \cdot J \)-similar to a diagonal block matrix \( \begin{pmatrix} D & O \\ O & O \end{pmatrix} \) where \( D \) is \( r \times r \) non singular, \( B \) is \( k \cdot J \cdot EP_r \), follows from lemma (1.10).

Conversely, if \( B = \begin{pmatrix} D & O \\ O & O \end{pmatrix} \) with \( D \) is \( r \times r \) non singular is \( k \cdot J \cdot EP_r \), the again by using lemma (1.10), \( kJ = k_1J_1k_2J_2 \) and \( K = \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix} \) and \( J = \begin{pmatrix} J_1 & O \\ O & J_2 \end{pmatrix} \). Since \( A \) is unitarily \( k \cdot J \)-similar to \( B = \begin{pmatrix} D & O \\ O & O \end{pmatrix} \) there exists unitary matrix \( U \) such that
\[ A = KJUJK \begin{pmatrix} D & O \\ O & O \end{pmatrix} U^* \]. Since \( B \) is \( k \cdot J \cdot EP_r \), then by theorem (1.4), \( KJB = KJ \begin{pmatrix} D & O \\ O & O \end{pmatrix} U^* KJA = EP_r \), \( KJA \)
is \( EP_r \). Now \( A \) is \( k \cdot J \cdot EP \) follows from theorem (1.4) and \( \text{rk}(A) = r \). Hence \( A \) is \( k \cdot J \cdot EP_r \).

**Theorem 1.14:**

If \( A \) is \( k \cdot J \cdot EP_r \) then \((\lambda, x)\) is a \( k \cdot J \)-eigen value, \( k \cdot J \)-eigen vector pair for \( A \iff (1/\lambda, kJ(x)) \) is a \( k \cdot J \)-eigen value, \( k \cdot J \)-eigen vector pair for \( A^\dagger \).

**Proof:**

We assume \((\lambda, x)\) is a \( k \cdot J \)-eigenvalue, \( k \cdot J \)-eigen vector pair of \( A \). \( \iff Ax = \lambda Kx \iff KAx = K^2 \lambda Ix \iff KAx = \lambda Ix \) [since \( k^2 = I \)] \( \iff KA^I x = \lambda J^2 x \) [since \( J^2 = I \)] \( \iff KJA^I x = \lambda x \iff (KJA)^I x = (1/\lambda) x \iff A^I (KJ)x = (1/\lambda) KJ(x) \iff (1/\lambda, KJx) \) is a \( k \cdot J \)-eigen value, \( k \cdot J \)-eigen vector pair for \( A^\dagger \). Hence the theorem.

**References:**