PAIR SUM LABELING OF GRAPH OPERATIONS
IN $\Gamma(Z_n)$

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ABSTRACT: In this chapter, we investigate the pair sum labeling behavior of the graphs $\Gamma(Z_{2p}) \cup \Gamma(Z_{2q})$, $\Gamma(Z_p) \cup \Gamma(Z_q)$, $\Gamma(Z_8) \cup \Gamma(Z_{2p})$, $\Gamma(Z_6) \cup \Gamma(Z_{2p})$, $m(G(p))$, $p \leq 5$, Comp $\Gamma(Z_8) \ast \Gamma(Z_4)$, $\Gamma(Z_6) \ast \Gamma(Z_4)$, $\Gamma(Z_9) \ast \Gamma(Z_4)$, $S(\Gamma(Z_{2p}))$, $S(\Gamma(Z_6) \ast \Gamma(Z_4))$, $S(\Gamma(Z_8) \ast \Gamma(Z_4))$ and $S(\Gamma(Z_9) \ast \Gamma(Z_4))$.

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1. INTRODUCTION

Let $G$ be a $(r, s)$ graph. An one to one map $f : V(G) \rightarrow \{\pm 1, \pm 2, \ldots, \pm r\}$ is called a pair sum labeling if the induced edge mapping, $f_e : E(G) \rightarrow Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm l_1, \pm l_2, \ldots, \pm l_{s/2}\}$ or $\{\pm l_1, \pm l_2, \ldots, \pm l_{(s-1)/2}\} \cup \{l_{(s+1)/2}\}$ according as $s$ is even or odd. A graph with a pair sum labeling defined on it is called pair sum graph. The pair sum labeling was introduced in [3] by R.Ponraj and et al. In [3], [4], [5] and [6] they study the pair sum labeling of cycle, path, star and some of their related graphs. Let $R$ be a commutative ring and let $Z(R)$ be its set of zero-divisors. We associate a graph $\Gamma(R)$ to $R$ with vertices $\Gamma(R)^* = Z(R) - \{0\}$, the set of non-zero zero divisors of $R$ and for distinct $u, v \in Z(R)^*$, the vertices $u$ and $v$ are adjacent if and only if $uv = 0$. The zero divisor graph is very useful to find the algebraic structures and properties of rings. The idea of a zero divisor graph of a commutative ring was introduced by I.Beck in [2]. The first simplification of Beck’s zero divisor graph was introduced by D.F.Anderson and P.S.Livinston [1]. Their motivation was to give a better illustration of the zero divisor structure of the ring. D.F.Anderson and P.S.Livinston, and others e.g., [7, 8, 9], investigate the interplay between the graph theoretic properties of $\Gamma(R)$ and the ring theoretic properties of $R$. Throught this paper, we consider the commutative ring $R$ by $Z_n$ and zero divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$.

2. PAIR SUM LABELING OF UNION OF $\Gamma(Z_n)$

Theorem 2.1. $\Gamma(Z_{2p}) \cup \Gamma(Z_{2q})$ is a pair sum graphs where $p$ and $q$ are different prime numbers.
Proof. Let \( \{p, 2, 4, \ldots, 2(p - 1)\} \) be the vertices of \( \Gamma(Z_{2p}) \). That is, the vertex set of \( \Gamma(Z_{2p}) \) is \( \{u, u_1, u_2, \ldots, u_{p-1}\} \) and the \( E(\Gamma(Z_{2p})) = \{uu_i : 1 \leq i \leq p\} \). Let \( \{q, 2, 4, \ldots, 2(q - 1)\} \) be the vertices of \( \Gamma(Z_{2q}) \), That is, the vertex set of \( \Gamma(Z_{2q}) \) is \( \{v, v_1, v_2, \ldots, v_{q-1}\} \) and \( E(\Gamma(Z_{2q})) = \{vv_i : 1 \leq i \leq q\} \), where \( p \) and \( q \) are distinct prime numbers.

Case (i): Let \( p = q \).

Clearly from [6], \( T \cup T \) is a pair sum tree, for any tree.

Case (ii): Without loss of generality, assume that \( q > p \).

Define,
\[
\begin{align*}
f(u) &= 1 \\
f(u_i) &= i + 1, 1 \leq i \leq n \\
f(v) &= -1 \\
f(v_i) &= -(i + 1), 1 \leq i \leq n
\end{align*}
\]

\( f(v_{p+1}) = (p + 3 + i), 1 \leq i \leq \frac{q - p - 1}{2} \), if \( q - p \) is even

\( f(v_{p+1}) = -(p + 3 + i), 1 \leq i \leq \frac{q - p - 1}{2} \), if \( q - p \) is odd

\( f(v_{p+2}) = p + i + 1, 1 \leq i \leq \frac{q - p}{2} \), if \( q - p \) is even

\( f(v_{p+2}) = p + i + 1, 1 \leq i \leq \frac{q - p - 1}{2} \), if \( q - p \) is odd

Thus the edge set,
\[
E(\Gamma(Z_{2p}) \cup \Gamma(Z_{2q})) = \{\pm 3, \pm 5, \ldots, \pm (p + 2)\} \cup \{\pm (p + 3), \ldots, \pm (p + q + 3)/2\} \cup \{-(p + q + 5)/2\}
\]

if \( q - n \) is odd. Clearly the function is a pair sum labeling. Hence, for any distinct prime numbers \( p \) and \( q \), \( \Gamma(Z_{2p}) \cup \Gamma(Z_{2q}) \) is a pair sum labeling graph.

**Theorem 2.2.** For any graph \( \Gamma(Z_n) \), the following holds:

(i) \( \Gamma(Z_8) \cup \Gamma(Z_{2p}) \) is a pair sum graph.

(ii) \( \Gamma(Z_6) \cup \Gamma(Z_{2p}) \) is a pair sum graph.

(iii) \( \Gamma(Z_4) \cup \Gamma(Z_{2p}) \) is a pair sum graph.

Proof. (i) To prove \( \Gamma(Z_8) \cup \Gamma(Z_{2p}) \) is a pair sum graph.

Let uvw be the path \( \Gamma(Z_8) \). Since the vertex set of \( \Gamma(Z_8) \) is \( \{2, 4, 6\} \), Clearly, \( \Gamma(Z_8) \) is isomorphic to \( P_3 \). Let \( V(\Gamma(Z_{2p})) = \{v, v_i : 1 \leq i \leq p\} \) and \( E(\Gamma(Z_{2p})) = \{vv_i : 1 \leq i \leq p\} \).

If \( p = 3 \), then \( \Gamma(Z_{2p}) = \Gamma(Z_6) \). Clearly, we know that \( \Gamma(Z_6) \) is isomorphic to \( K_{1,2} \) or \( \Gamma(Z_6) \) is isomorphic to \( P_3 \). Hence, the union of two paths (or) the union of trees is pair sum graph. That is \( \Gamma(Z_8) \cup \Gamma(Z_{2p}) \) is a pair sum graph.

(ii) To prove \( \Gamma(Z_6) \cup \Gamma(Z_{2p}) \) is a pair sum graph.

Let uvw be the path \( \Gamma(Z_6) \). Since the vertex set of \( \Gamma(Z_6) \) is \( \{2, 3, 4\} \), Clearly, \( \Gamma(Z_6) \) is isomorphic to \( P_3 \). Using the above proof (i) \( \Gamma(Z_6) \cup \Gamma(Z_{2p}) \) is a pair sum graph.

(iii) To prove \( \Gamma(Z_4) \cup \Gamma(Z_{2p}) \) is a pair sum graph.
Let uv be the path $\Gamma(Z_6)$. The vertex set of $\Gamma(Z_6)$ is $\{3,6\}$. Clearly, $\Gamma(Z_9)$ is a path of order 2. We know that the union of two trees is a pair sum labeling graph. Hence, $\Gamma(Z_9) \cup \Gamma(Z_{2p})$ is a pair sum labeling.

In general, for any path $P_m$ in $\Gamma(Z_p)$, the vertex set of $P_m$ is $\{u_1,u_2,\ldots,u_m\}$. That is $u_1, u_2, \ldots, u_m$ be the path $P_m$. Let $V(\Gamma(Z_{2p})) = \{v,v_i : 1 \leq i \leq p\}$ and $E(\Gamma(Z_{2p})) = \{vv_i : 1 \leq i \leq p\}$.

Case (i): $m = p$
Define,
f(u_i) = i, 1 \leq i \leq m
f(v) = -1
Hence, $f(E(P_m) \cup \Gamma(Z_{2p})) = \{\pm 3, \pm 5, \ldots, \pm (2p - 1)\} \cup \{- (2p + 1)\}$

Case (ii): $p > m$
Define,
f(u_i) = i, 1 \leq i \leq m
f(v) = -1
f(v_i) = -2i, 1 \leq i \leq m - 1
f(v_{m+2i-1}) = 2m + i, 1 \leq i \leq \frac{p-m+1}{2}, \text{if } p - m \text{ is odd}
f(v_{m+2i-1}) = 2m + i, 1 \leq i \leq \frac{p-m}{2}, \text{if } p - m \text{ is even}

Here
$f(E(P_m) \cup \Gamma(Z_{2p})) = \{\pm 3, \pm 5, \ldots, \pm (2m - 1)\} \cup \{\pm 2m, \pm (2m - 1), \ldots, \frac{(3m + p - 2)}{2}\}$ if $p - m$ is odd
$f(E(P_m) \cup \Gamma(Z_{2p})) = \{\pm 3, \pm 5, \ldots, \pm (2m - 1)\} \cup \{\pm 2m, \pm (2m + 1), \ldots, \frac{(3m + p - 2)}{2}\} \cup \{\frac{(3m + n)}{2}\}$ if $n - m$ is even. Then, f is a pair sum labeling.

**Theorem 2.3.** If any prime $p \leq 5$, then $m\Gamma(Z_p)$ is a pair sum graph.

**Theorem 2.4.** If $p \geq 11$, then $m\Gamma(Z_p)$ is not a pair sum graph.

Proof. We prove this by the method of contradiction. Suppose, $m\Gamma(Z_p)$ is a pair sum graph. We know that, if $\Gamma(Z_n)$ is a (r,s) pair sum graph then $s \leq 4r - 2$, where r is the number of vertices and s is the number of edges in $\Gamma(Z_n)$. We know that $\Gamma(Z_p)$ is isomorphic with $K_{p-1}$. Then, the number of edges in a complete graph $K_{p-1}$ is $\frac{(p-1)(p-2)}{2}$. Then the total edges of m copies of $\Gamma(Z_p)$ is $m\left(\frac{(p-1)(p-2)}{2}\right)$.
Then, \( \frac{m(p-1)(p-2)}{2} \leq 4(p-1) - 2 = 4p - 6 \)
\( m(p-1)(p-2) - 8p + 12 \leq 0 \)
\( 8m - m(p-1)(p-2) - 12 \geq 0 \)
If \( p = 11 \) and \( m = 5 \), then \( 8m - m(p-1)(p-2) - 12 \) is a negative value. Clearly,
\( 8m - m(p-1)(p-2) - 12 \geq 0 \), a contradiction.
Hence, If any prime \( p \geq 11 \), then \( m\Gamma(Z_p^3) \) is not a pair sum graph.

3. PAIR SUM LABELING OF CORONA OF TWO ZERO DIVISOR GRAPHS
In this section, we investigate the pair sum labeling behavior of some graphs obtained as a Corona of
two standard graph in zero divisor graphs.

**Theorem 3.1.** (i) The comb \( \Gamma(Z_6) \square \Gamma(Z_4) \) is a pair sum graph.
(ii) \( \Gamma(Z_8) \square \Gamma(Z_4) \) is a pair sum graph.
(iii) \( \Gamma(Z_9) \square \Gamma(Z_4) \) is a pair sum graph.

Proof. (i) and (ii), We know that, \( \Gamma(Z_6) \square \Gamma(Z_4) \) and \( \Gamma(Z_8) \square \Gamma(Z_4) \) are same graphs. Since \( \Gamma(Z_6) \) and \( \Gamma(Z_8) \) are isomorphic to \( P_3 \).
Let \( P_3 \) be the path \( uvw \) and \( u \) and \( w \) are the pendent vertices, adjacent to \( v \). Let \( n = 3 = 2m + 1 \),
where \( m = 1 \).
Define, \( f : V(P_3 \square \Gamma(Z_4)) \rightarrow \{ \pm 1, \pm 2, ..., \pm (2m+1) \} \) by
\( f(u_i) = 2i; 1 \leq i \leq m \)
\( f(w_i) = 2i - 4(m + 1); m + 1 \leq i \leq 2m + 1 \)
\( f(v_i) = 2i - 1; 1 \leq i \leq m \)
\( f(v_i) = 2m + 8; i = m + 1 \)
\( f(v_i) = 2i - 4(m + 3); m + 2 \leq i \leq 2m + 1 \)

Here, \( f(E(P_3) \square \Gamma(Z_4)) = \{ \pm 3, \pm 6, \pm 7, ..., \pm (4m - 2), \pm (4m - 1) \} \cup \{ \pm 4, \pm (4m - 4) \} \)
Then, \( f \) is a pair sum labeling.
(iii) We know that \( \Gamma(Z_9) \square \Gamma(Z_4) \) is isomorphic to \( P_2 OS K_1 \).
Let \( n = 2m = 2 \). Then define \( f : V(\Gamma(Z_9) \square \Gamma(Z_4)) \rightarrow \{ \pm 1, \pm 2 \} \) by \( n \equiv 2(\text{mod}4) \)
\( f(u_i) = 2i; if \ i \ is \ odd \ and \ i \leq m - 1 \)
\( f(u_i) = 2i - 1; if \ i \ is \ even \ and \ i \leq m \)
\( f(u_i) = -(2m + 1), if i = m \)
\( f(u_i) = 2m + 3, if i = m + 1 \)
\( f(u_i) = 2i - 1 - 4m, if \ i \ is \ odd \ m + 1 < i \leq 2m \)
\( f(u_i) = 2i - (4m + 2), if \ i \ is \ even \ m + 1 < i \leq 2m \)
\( f(v_i) = 2i - 1, if \ i \ is \ odd \ and \ i \leq m \)
\( f(v_i) = 2i, if \ i \ is \ even \ and \ i \leq m \)
\( f(v_i) = 2i - 1 - 4m, if \ i \ is \ even \ and \ m < i \leq 2m \)
\( f(v_i) = 2i - (4m + 2), if \ i \ is \ odd \ m < i \leq 2m \)
Here, \( f_\epsilon(E(\Gamma(Z_n)\cup \Gamma(Z_4))) = \{\pm 3, \pm 5, \ldots, \pm (4m - 5)\} \cup \{\pm 2, \pm 4, \pm 6\} \).

Then, \( f \) is a pair sum labeling. Therefore \( \Gamma(Z_6)\cup \Gamma(Z_4) \), \( \Gamma(Z_8)\cup \Gamma(Z_4) \) and \( \Gamma(Z_9)\cup \Gamma(Z_4) \) are pair sum graphs.

**Theorem 3.2.** The graph \( \Gamma(Z_n)\cup \Gamma(Z_4) \) is a pair sum graph, where \( n = 6, 8 \) and \( 9 \).

**Proof.** We know that \( \Gamma(Z_6) \cong \Gamma(Z_8) \cong \Gamma(Z_9) \cong \Gamma(Z_4) \). Let \( v_i \) and \( w_i \) be the pendent vertices adjacent to \( u \), for \( 1 \leq i \leq n \).

- **Case (i) \( n = 3 \) [odd]**
  
  Define,
  
  \[
  f(u_i) = 3i - 1; 1 \leq i \leq \frac{n-1}{2} \\
  f(u_i) = \frac{1-3n}{2}; i = \frac{n+1}{2} \\
  f(u_i) = 3i - (3n + 2); \frac{n+3}{2} \leq i \leq n \\
  f(v_i) = 3i; 1 \leq i \leq \frac{n-1}{2} \\
  f(v_i) = \frac{3n+5}{2}; i = \frac{n+1}{2} \\
  f(v_i) = 3i - (3n + 3); \frac{n+3}{2} \leq i \leq n \\
  f(w_i) = 3n - 5; i = 1 \\
  f(w_i) = \frac{3(n+1)}{2}; i = \frac{n+1}{2} \\
  f(w_i) = 3i - (3n + 1); \frac{n+3}{2} \leq i \leq n \\
  
  Here, \( f_\epsilon(E(\Gamma(Z_n)\cup \Gamma(Z_4))) = \{\pm 3, \pm 5, \ldots, \pm (3n - 4)\} \cup \{\pm (3n - 3)\} \).
  
  Then, \( f \) is a pair sum labeling.

- **Case (ii):** We know that \( \Gamma(Z_9) \cong \Gamma(Z_4) \). Clearly, \( u \) and \( v \) are pendent vertices in \( P_2 \). Let \( n = 2 \) is even.

  Since, \( P_2 \cup \Gamma(Z_4) = B_{2,2} \). Let \( V(B_{2,2}) = \{u, v, u_i, v_j : 1 \leq i \leq 2, 1 \leq j \leq 2\} \) and
  
  \[
  E(B_{2,2}) = \{uv, uu_i, vv_j : 1 \leq i \leq 2, 1 \leq j \leq 2\}. 
  
  Define, \( f : V(B_{2,2}) \to \{\pm 1, \pm 2, \ldots, \pm (2n + 2)\} \) by
  
  \[
  f(u) = -1 \\
  f(v) = 2 \\
  f(u_i) = -2i; 1 \leq i \leq n \\
  f(v_i) = 2i - 1; 1 \leq i \leq n, \text{ where } n = 2. 
  
  Thus, \( f_\epsilon(E(B_{2,2})) = \{\pm 3, \pm 5, \ldots, \pm (2n + 1)\} \cup \{1\} \) and have \( B_{2,2} \) is a pair sum graph. That is \( \Gamma(Z_9)\cup \Gamma(Z_4) \) is a pair sum graph.

4. **PAIR SUM LABELING ON SUBDIVISION GRAPHS IN \( \Gamma(Z_n) \)**
Hence, we investigate the pair sum labeling behavior of graphs obtained as the subdivision of some standard graphs in $\Gamma(Z_n)$.

**Theorem 4.1.** $S(\Gamma(Z_{2p}))$ is a pair sum graph.

**Proof.** Let $V(S(\Gamma(Z_{2p}))) = \{u, u_i, v_i : 1 \leq i \leq p - 1\}$ and $E(S(\Gamma(Z_{2p}))) = \{uu_i, uu_i, v_i : 1 \leq i \leq p - 1\}$.

Define $f : S(\Gamma(Z_{2p})) \to \{\pm 1, \pm 2, \ldots, \pm (2p - 1)\}$ by

$f(u) = -1$

$f(u_i) = i + 1 ; i = 1, 2, \ldots, n$

$f(v_i) = -(2i + 1) ; i = 1, 2, \ldots, n$

Then $f$ gives a pair sum labeling for $S(\Gamma(Z_{2p}))$.

**Theorem 4.2.** $S(\Gamma(Z_n) \circ \Gamma(Z_4))$ is a pair sum graph, where $n = 6, 8$ and $9$.

**Proof.** Let $V(S(\Gamma(Z_n) \circ \Gamma(Z_4))) = \{u : 1 \leq i \leq 2n - 1\} \cup \{w_i, v_i : 1 \leq i \leq n\}$.

Let $E(S(\Gamma(Z_n) \circ \Gamma(Z_4))) = \{uu_i : 1 \leq i \leq 2n - 2\} \cup \{u_{2i-1}w_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq n\}$.

Since, $\Gamma(Z_2)$ and $\Gamma(Z_4)$ is path with length 2, and $\Gamma(Z_9)$ is a path with length 1.

**Case (i):** $n$ is even $[\Gamma(Z_n) \cong P_2]$.

Since we know that any path is a pair sum graph. So, the Subdivision of $\Gamma(Z_n) \circ \Gamma(Z_4)$ is a pair sum graph.

**Case (ii):** $n = 3$ is odd $[\Gamma(Z_n) \cong \Gamma(Z_9) \cong P_1]$.

Define $f : V(S(P_3 \circ \Gamma(Z_4))) \to \{\pm 1, \pm 2, \ldots, \pm (4n - 1)\}$ by

$f(u_{(n+1)/2}) = 1$

$f(u_{n/2}) = 8$

$f(u_{(n+3)/2}) = -8$

$f(u_{n/2}) = 8$

$f(u_{(n-1)/2}) = -10i + 1, 1 \leq i \leq \left[\frac{n}{2}\right] - 1$

$f(u_{(n+1)/2}) = 5i + 5, 1 \leq i \leq \left[\frac{n}{2}\right] - 1$

$f(u_{(n+3)/2}) = -10i - 1, 1 \leq i \leq \left[\frac{n}{2}\right] - 1$

$f(u_{(n+1)/2}) = -(5i + 5), 1 \leq i \leq \left[\frac{n}{2}\right] - 1$

$f(w_{(n+1)/2}) = -2$

$f(w_{n/2}) = -5$

$f(w_{(n-1)/2}) = 5i + 7, 1 \leq i \leq \left[\frac{n}{2}\right] - 1$

$f(w_{(n+3)/2}) = -(5i + 7), 1 \leq i \leq \left[\frac{n}{2}\right] - 1$

$f(v_{(n+1)/2}) = 3$
\[ f(v_{(n-1)/2}) = 9 \]
\[ f(v_{(n+3)/2}) = -9 \]
\[ f(v_{(n-1)/2-i}) = 5i + 8, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \]
\[ f(v_{(n+3)/2+i}) = -(5i + 8), 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \]

Here \( f_e(E(S(P_3 \Theta \Gamma(Z_4)))) = \{\pm 1, \pm 2, ..., \pm 5\} \)
Then \( f \) is pair sum labeling.

5. PAIR SUM LABELING OF PATH AND CYCLE RELATED TO ZERO DIVISOR GRAPH

In this chapter, we prove that \( \Gamma(Z_2) \times \Gamma(Z_2) \), \( C_n \times \Gamma(Z_2) \) and \( [C_m, P_n] \) are pair sum graphs.

**Theorem 5.1.** The graph \( \Gamma(Z_{2^n}) \times \Gamma(Z_{2^n}) \) is a pair sum graph.

**Theorem 5.2.** The graph \( C_n \times \Gamma(Z_2) \) is a pair sum graph, if \( n \) is even.

**Proof.** Let \( V(C_n \times \Gamma(Z_2)) = \{ u_i, v_i : 1 \leq i \leq n \} \) and \( E(C_n \times \Gamma(Z_2)) = \{ u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n \} \cup \{ u_i, v_i : 1 \leq i \leq n \} \cup \{ u_n, u_1, v_n, v_1 \} \).

Define \( f: V(C_n \times \Gamma(Z_2)) \to \{\pm 1, \pm 2, ..., \pm n\} \) as follows

Case (i) \( n = 4m + 2 \).
Define, \( f(u_i) = i; 1 \leq i \leq 2m + 1 \)
\[ f(v_i) = 8m - 2i + 6; 1 \leq i \leq 2m + 1 \]
\[ f(v_{2m+1+i}) = -8m + 2i - 6; 1 \leq i \leq 2m + 1 \]

Here \( f_e(E(C_n \times \Gamma(Z_2))) = \{\pm 3, \pm 5, ..., \pm (4m + 1)\} \cup \{\pm 2m\} \cup \{\pm (6m + 5), \pm (6m + 6), ..., \pm (8m + 5)\} \).

Case (ii) \( n = 4m \).
Define, \( f(u_i) = i; 1 \leq i \leq 2m - 1 \)
\[ f(u_{2m+i}) = -i; 1 \leq i \leq 2m - 1 \]
\[ f(u_{4m}) = -(2m + 1) \]
\[ f(v_{2m+i}) = 8m - 2i + 2; 1 \leq i \leq 2m \]
\[ f(v_{2m+i}) = -(4m + 2i); 1 \leq i \leq 2m \]

Here \( f_e(E(C_n \times \Gamma(Z_2))) \)
\[ = \{\pm 3, \pm 5, ..., \pm (4m - 3)\} \cup \{\pm 2m, \pm 4m\} \cup \{\pm (4m + 3), \pm (4m + 6), ..., \pm (10m - 3)\} \cup \{\pm (10m + 1)\} \]

Clearly, \( f \) is a pair sum labeling.

**Theorem 5.3.** The graph \( [C_m, \Gamma(Z_2)] \) is a pair sum graph.

**Proof.** Let the first copy of the cycle \( C_m \) be \( u_1 u_2 ..., u_m u_1 \) and second copy of cycle \( C_m \) be \( v_1 v_2 ..., v_m v_1 \). Let \( \Gamma(Z_2) \) be path \( P_3 \), \( w_1 w_2 w_3 \). Let \( V([C_m, \Gamma(Z_2)]) = V(C_m) \cup V(C_m) \cup V(\Gamma(Z_2)) \) and \( E([C_m, \Gamma(Z_2)]) = E(C_m) \cup E(C_m) \cup E(\Gamma(Z_2)) \cup \{u_1 w_1, w_3 v_1\} \).

Define \( f: V([C_m, \Gamma(Z_2)]) \to \{\pm 1, \pm 2, ..., \pm 3m\} \) by
\[ f(w_1) = 2 \]
\[ f(w_2) = 1 \]
\[ f(w_3) = -4 \]
f (v_i) = -3m - i + 1, 1 \leq i \leq m \\
f (u_i) = 2m + 2 + i, 1 \leq i \leq m - 2 \\
f (u_{m-1}) = 2m + 1 \\
f (u_m) = 2m + 2 \\

Here,
\[ f_r (E([C_n, \Gamma(Z_6)])) = \{ \pm 3, \pm 5, ..., \pm n \} \cup \{ \pm (4m+3), \pm (4m+5), ..., \pm (6m-1) \} \cup \{ \pm (5m+1), \left( \frac{5m+7}{2} \right) \} \]

Then, Clearly \( f \) is a pair sum labeling.

REFERENCES