PARTIAL SUMS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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Abstract: Let $f_m(z) = z + \sum_{k=2}^{m} a_k z^k$ be the sequence of partial sums of a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ that is analytic in $|z| < 1$ and belong to the class $S_\alpha$, where $0 \leq \alpha < 1$. When the coefficients of $a_k$ are "small" we determine sharp lower bounds for $\text{Re} \left\{ \frac{D^p f(z)}{D^p f_m(z)} \right\}$ and $\text{Re} \left\{ \frac{D^p f_m(z)}{D^p f(z)} \right\}$, where $D^p$ stands for the Salagean derivative introduced in [4].

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I. INTRODUCTION

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disc $U = \{ z : |z| < 1 \}$. Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $U$. Further $T$ denotes subclass of $A$ consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0.$$

We denote by $S^*\alpha, K(\alpha), (0 \leq \alpha < 1)$, the class of starlike functions of order $\alpha$ and class of convex functions of order $\alpha$, respectively, where

$$S^*\alpha = \left\{ f \in S : \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, z \in U \right\},$$

and

$$K(\alpha) = \left\{ f \in S : \text{Re} \left\{ 1 + \frac{zf^*(z)}{f'(z)} \right\} > \alpha, z \in U \right\}.$$

We also denote by $T^*\alpha$ and $C(\alpha)$ the subclasses of $T$ that are, respectively, starlike of order $\alpha$ and convex of order $\alpha$.

For $f(z)$ belonging to $A$, Salagean [4] has introduced the following operator called the Salagean operator

$$D^p f(z) = f(z)$$
\[ D^1 f(z) = zf'(z) \]
\[ D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1,2,3,\ldots\}). \]

Note that
\[ D^n f(z) = z + \sum_{k=2}^{\infty} k^\alpha a_k z^k, \quad (n \in \mathbb{N}_0 = N \cup \{0\}). \]

A function \( f(z) \in A \) is said to belong to the class \( S_n(\alpha) \) if it satisfies
\[ \Re \left( \frac{D^{n+1} f(z)}{D^n f(z)} \right) > \alpha, \quad (z \in U) \]
for some \( 0 \leq \alpha < 1 \) and \( n \in \mathbb{N}_0 \).

The class \( S_n(\alpha) \) have been studied by various authors (for example [1], [2], [3]).

Note that
\[ S_0(\alpha) = S^*(\alpha) \]
and
\[ S(\alpha) = K(\alpha). \]

A sufficient condition for a function \( f \) of the form (1.1) to be in \( S_n(\alpha) \) is that
\[ \sum_{k=2}^{\infty} \frac{k^\alpha (k-\alpha)}{1-\alpha} a_k \leq 1. \]
(1.4)

For the functions of the form (1.2) the sufficient condition (1.4) is also necessary. For detailed study see [1].

In the present paper, we determine sharp lower bounds for \( \Re \left( \frac{D^n f(z)}{D^n f_m(z)} \right) \) and \( \Re \left( \frac{D^p f_m(z)}{D^p f(z)} \right) \) (where \( f_m(z) = z + \sum_{k=2}^{m} a_k z^k \) is the sequence of partial sums of \( f(z) \) given by (1.1) and coefficients of \( f \) are sufficiently small to satisfy the condition (1.4)) which are motivated from the investigation of Silverman [5].

2. Main Results

**Theorem 2.1:** If \( f \) of the form (1.1) satisfies the condition (1.4), then
\[ \Re \left( \frac{D^n f(z)}{D^n f_m(z)} \right) \geq \frac{(m+1)^{\alpha-p} (m+1-\alpha)-(1-\alpha)}{(m+1)^{\alpha-p} (m+1-\alpha)}, \quad (z \in U). \]
(2.1)

and
\[ \Re \left( \frac{D^p f_m(z)}{D^p f(z)} \right) \geq \frac{(m+1)^{\alpha-p} (m+1-\alpha)}{(m+1)^{\alpha-p} (m+1-\alpha)+(1-\alpha)}, \quad (z \in U). \]
(2.2)

The results (2.1) and (2.2) are sharp for every \( m \) with the function given by
\[ f(z) = z + \frac{1-\alpha}{(m+1)^{\alpha} (m+1-\alpha)} z^{m+1}. \]
(2.3)

where \( 0 \leq \alpha < 1, n \in \mathbb{N}_0 \) and \( p \leq n+1 \).

**Proof:** Define the function \( \omega(z) \) by
\[
1 + \omega(z) = \frac{(m+1)^{n-p} (m+1-\alpha)}{1-\alpha} \left[ D^p f(z) \frac{(m+1)^{n-p} (m+1-\alpha) - (1-\alpha)}{D^p f_m(z)} \right] \\
= 1 + \sum_{k=2}^{\infty} k^p a_k z^{k-1} + \frac{(m+1)^{n-p} (m+1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p a_k z^{k-1} \\
= 1 + \sum_{k=2}^{\infty} k^p a_k z^{k-1}.
\]  

(2.4)

It suffices to show that \(|\omega(z)| \leq 1\). Now, from (2.4) we can write

\[
\omega(z) = \frac{(m+1)^{n-p} (m+1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p a_k z^{k-1}.
\]

Hence we obtain

\[
|\omega(z)| \leq \frac{(m+1)^{n-p} (m+1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p |a_k|.
\]

Now \(|\omega(z)| \leq 1\) if

\[
2 \frac{(m+1)^{n-p} (m+1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p |a_k| \leq 2 - \sum_{k=2}^{m} k^p |a_k|.
\]

or, equivalently,

\[
\sum_{k=2}^{m} k^p |a_k| + \frac{(m+1)^{n-p} (m+1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p |a_k| \leq \sum_{k=2}^{\infty} k^p (k-\alpha) |a_k|.
\]

(2.5)

From the condition (1.4), it is sufficient to show that

\[
\sum_{k=2}^{m} k^p |a_k| + \frac{(m+1)^{n-p} (m+1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p |a_k| \leq \sum_{k=2}^{\infty} k^p (k-\alpha) |a_k|
\]

which is equivalent to

\[
\sum_{k=2}^{m} k^{n+1} - \alpha k^n - k^p (1-\alpha) |a_k| + \sum_{k=m+1}^{\infty} k^{n+1} - \alpha k^n - (m+1)^{n-p} (m+1-\alpha) k^p |a_k| \geq 0.
\]

To see that the function given by (2.3) gives the sharp result, we observe that for \(z = r e^{i\gamma}\) that

\[
\frac{D^p f(z)}{D^p f_m(z)} = 1 + \frac{1-\alpha}{(m+1)^p (m+1-\alpha)} (m+1)^p z^m \to 1 - \frac{1-\alpha}{(m+1)^{n-p} (m+1-\alpha)}
\]

\[
= \frac{(m+1)^{n-p} (m+1-\alpha) - (1-\alpha)}{(m+1)^{n-p} (m+1-\alpha)}, \quad \text{when } r \to 1.
\]

To prove the second part of this theorem, we may write
\[
1 + \omega(z) = \frac{(m+1)^{n-p}(m+1-\alpha)+(1-\alpha)}{1-\alpha} \left[ D^n f_m(z) - \frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha} \right]
\]
\[
= 1 + \sum_{k=2}^{m} k^p a_k z^{k-1} - \frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p a_k z^{k-1}
\]

where
\[
|\omega(z)| \leq \frac{(m+1)^{n-p}(m+1-\alpha)+(1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p |a_k| \leq 1.
\]

This last inequality is equivalent to
\[
\sum_{k=2}^{m} k^p |a_k| + \frac{(m+1)^{n-p}(m+1-\alpha)}{1-\alpha} \sum_{k=m+1}^{\infty} k^p |a_k| \leq 1.
\]

Since the L.H.S. of (2.6) is bounded above by \( \sum_{k=2}^{m} k^p (k-\alpha) \sum_{k=m+1}^{\infty} k^p |a_k| \), and the proof is complete. Finally, equality holds in (2.2) for the function given in (2.3).

Taking \( n = 0, p = 0 \) in Theorem 2.1, we obtain the following result given by Silverman in [5].

Corollary 2.2 ([5]). If \( f \) of the form (1.1) and satisfies the condition \( \sum_{k=2}^{m} k^{-\alpha} (k-\alpha) \sum_{k=m+1}^{\infty} k^{-\alpha} |a_k| \leq 1 \), then
\[
\text{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \geq \frac{m}{m+1-\alpha}, \quad (z \in U).
\]
and
\[
\text{Re} \left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{m+1-\alpha}{m+2-2\alpha}, \quad (z \in U).
\]

The results are sharp with the function given by
\[
f(z) = z + \frac{1-\alpha}{m+1-\alpha} z^{m+1}.
\]

Taking \( n = 0, p = 1 \) in Theorem 2.1, we obtain the following result given by Silverman in [5].

Corollary 2.3 ([5]) If \( f \) of the form (1.1) and satisfies the condition \( \sum_{k=2}^{m} k^{-\alpha} (k-\alpha) \sum_{k=m+1}^{\infty} k^{-\alpha} |a_k| \leq 1 \), then
\[
\text{Re} \left\{ \frac{f'(z)}{f_m'(z)} \right\} \geq \frac{m\alpha}{m+1-\alpha}, \quad (z \in U).
\]
and
\[
\text{Re} \left\{ \frac{f_m'(z)}{f'(z)} \right\} \geq \frac{m+1-\alpha}{(m+1)(2-\alpha)-\alpha}, 
\quad (z \in U). 
\]  
(2.11)

The results are sharp with the function given by (2.9).

Taking \( n = 1, p = 0 \) in Theorem 2.1, we obtain the following result given by Silverman in [5].

**Corollary 2.4** ([5]) If \( f \) of the form (1.1) and satisfies the condition \( \sum_{k=2}^{\infty} k(k-\alpha) |d_k| \leq 1 \), then

\[
\text{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \geq \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}, 
\quad (z \in U), 
\]  
(2.12)

and

\[
\text{Re} \left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{(m+1)(m+1-\alpha)}{(m+1)(m+1-\alpha)+(1-\alpha)}, 
\quad (z \in U). 
\]  
(2.13)

The results are sharp with the function given by

\[
f(z) = z + \frac{1-\alpha}{(m+1)(m+1-\alpha)} z^{m+1}
\]  
(2.14)

Taking \( n = 1, p = 1 \) in Theorem 2.1, we obtain the following result given by Silverman in [5].

**Corollary 2.5** ([5]) If \( f \) of the form (1.1) and satisfies the condition \( \sum_{k=2}^{\infty} k(k-\alpha) |d_k| \leq 1 \), then

\[
\text{Re} \left\{ \frac{f'(z)}{f_m'(z)} \right\} \geq \frac{m}{m+1-\alpha}, 
\quad (z \in U), 
\]  
(2.15)

and

\[
\text{Re} \left\{ \frac{f_m'(z)}{f'(z)} \right\} \geq \frac{m+1-\alpha}{m+2-2\alpha}, 
\quad (z \in U). 
\]  
(2.16)

The results are sharp with the function given by (2.14).

**REFERENCES**


