K-Distance Non-negative Signed Domination Number of Graphs

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Abstract: Let G be a finite and simple graph with the vertex set \( V = V(G) \) and edge set \( E = E(G) \). If \( v \) is a vertex of a graph \( G \), the open \( k \)-neighborhood of \( v \), denoted by \( N_k(v) \) and \( N_k[v] = N_k(v) \cup \{v\} \) is the closed \( k \)-neighborhood of \( v \). A function \( f: V(G) \rightarrow \{-1, +1\} \) is a distance non-negative signed dominating function (k-DNNSDF) of a graph \( G \), if for every vertex \( v \in V \), \( f(N_k[v]) = \sum_{u \in N_k[v]} f(u) \geq 0 \). The k-distance non-negative signed domination number (k-DNNSDN) of a graph \( G \) equals the minimum weight of a k-DNNSDF of \( G \), denoted by \( \gamma_{k,NN}^N(G) \). This paper contains some properties of k-DNNSDN in graphs and some families of graphs such as cycles, paths, complement of cycles, complete graphs, wheel graphs and friendship graphs which admit 2-DNNSDF.

Index Terms - Signed dominating function, k-distance non-negative signed dominating function.

I. INTRODUCTION

Let \( G \) be a finite and simple graph with the vertex set \( V = V(G) \) and edge set \( E = E(G) \). If \( v \) is a vertex of a graph \( G \), the open \( k \)-neighborhood of \( v \), denoted by \( N_k(v) \) and \( N_k[v] = N_k(v) \cup \{v\} \) is the closed \( k \)-neighborhood of \( v \). A function \( f: V(G) \rightarrow \{-1, +1\} \) is a signed dominating function of \( G \), if for every vertex \( v \in V \), \( f(N[v]) \geq 1 \). The signed domination number, denoted by \( \gamma_s(G) \), is the minimum weight of a signed dominating function on \( G \) [1].

In 2013 [2], Zhongsheng Huang et al. introduced the concept of on non-negative signed domination in graphs. A function \( f: V(G) \rightarrow \{-1, +1\} \) is a non-negative signed dominating function of \( G \), if for every vertex \( v \in V \), \( f(N[v]) \geq 0 \). The non-negative signed domination number, denoted by \( \gamma_{NN}^N(G) \), is the minimum weight of a non-negative signed dominating function on \( G \).

In this paper, we introduced the concept of k-distance non-negative signed dominating function. A function \( f: V(G) \rightarrow \{-1, +1\} \) is a k-distance non-negative signed dominating function (k-DNNSDF) of a graph \( G \), if for every vertex \( v \in V \), \( f(N_k[v]) = \sum_{u \in N_k[v]} f(u) \geq 0 \). The k-distance non-negative signed domination number (k-DNNSDN) of a graph \( G \) equals the minimum weight of a k-DNNSDF of \( G \), denoted by \( \gamma_{k,NN}^N(G) \). This paper contains some properties of k-DNNSDN in graphs and some families of graphs such as cycles, paths, complement of cycles, complete graphs, wheel graphs and friendship graphs which admit 2-DNNSDF.

MAIN RESULTS

In this section, we obtain some properties of k-DNNSDN in graphs.

Lemma 1. Let \( f \) be a k-DNNSDF of \( G \) and let \( S \subseteq V \). Then \( f(S) \equiv |S|(\text{mod} \ 2) \).

Proof. Let \( S^+ = \{v|f(v) = 1, v \in S\} \) and \( S^- = \{v|f(v) = -1, v \in S\} \). Then \( |S^+| + |S^-| = |S| \) and \( |S^+| - |S^-| = f(S) \). Therefore \( f(S) + |S| = 2|S| \).

Theorem 1. Let \( G \) be a graph of order \( n \). If \( \gamma_{k,NN}^N(G) = n \), then \( G \cong \overline{K_n} \).

Proof. Proof. Let \( \gamma_{k,NN}^N(G) = n \). If \( \deg(v) \geq 1 \) for some \( v \in V(G) \), then the function \( f: V(G) \rightarrow \{-1, +1\} \) defined by \( f(v) = -1 \) and \( f(x) = +1 \) for all other vertices \( x \), is k-DNNSDF and this implies that \( \gamma_{k,NN}^N(G) \leq n - 2 \), a contradiction. Thus \( \Delta(G) = 0 \) and so \( G \cong \overline{K_n} \).

Observation 2.1. Let \( G \) be a graph of order \( n \) and \( k \) be a positive integer. Then \( \gamma_{k,NN}^N(G) = \gamma_{s}^N(G^k) \).
Proof. Let \( f \) be a k-DNNSDF of \( G \). It is easy to see that for every \( v \in V(G) \), \( N[v] = N_{G^k}[v] \). Hence \( f(N_{G^k}[v]) = f(N_k[v]) \) Therefore \( f \) is a k-DNNSDF of \( G \) if and only if \( f \) is a k-distance non-negative signed dominating set of \( G^k \). Thus \( \gamma^N_{2,k}(G) = \gamma^N_{2,k}(G^k) \).

**Lemma 2.** Let \( G \) be a graph of order \( n \). Then \( 2\gamma(G) - n \leq \gamma^N_{2}(G) \).

Proof. Let \( f \) be a minimum non-negative signed dominating function of \( G \). Let \( V^+ = \{ u \in V : f(u) = +1 \} \) and \( V^- = \{ u \in V : f(u) = -1 \} \). If \( v \in V^- \) since \( f(N_{G^k}[v]) \geq 0 \), then \( v \) has at least one neighbor in \( V^+ \). Therefore \( V^+ \) is a dominating set for \( G \) and \( |V^+| \geq \gamma(G) \). Since \( \gamma^N_{2}(G) = |V^+| - |V^-| \) and \( n = |V^+| + |V^-| \), then \( \gamma^N_{2}(G) = 2|V^+| - n \) and finally we have \( \gamma^N_{2}(G) \geq 2\gamma(G) - n \).

**Lemma 3.** Let \( n \geq 5 \) be an integer. Then the cycle \( C_n \) admits 2-DNNSDF with

\[
\gamma^N_{2,2}(C_n) \leq k \text{ when } n = 5k.
\]

\[
\gamma^N_{2,2}(C_n) \leq k + 1 \text{ when } n = 5k+1.
\]

\[
\gamma^N_{2,2}(C_n) \leq k + 2 \text{ when } n = 5k + 2 \text{ or } n = 5k + 4.
\]

\[
\gamma^N_{2,2}(C_n) \leq k + 3 \text{ when } n = 5k + 3.
\]

Proof. Let \( n \geq 5 \) be an integer. Let \( V(C_n) = \{ a_i / 1 \leq i \leq n \} \) and \( E(C_n) = \{ a_i a_{i+1} / 1 \leq i \leq n \} \). Define a function \( f : V(C_n) \rightarrow \{-1,+1\} \) such that \( f(a_i) = -1 \) when \( i = 5l \) or \( i = 5l - 1 \), \( l \geq 1 \) and otherwise \( f(a_i) = +1 \).

Consider the vertex \( a_i \) for \( 1 \leq i \leq n, N_2[a_i] = \{ a_{i-2}, a_{i-1}, a_i, a_{i+1}, a_{i+2} \} \). From the above labeling, it is easy to observe that at least 3 vertices of any five consecutive vertices must have +1 sign and hence \( f(N_2[a_i]) \geq 1 \) for all \( i, 1 \leq i \leq n \).

Thus from the above labeling the result follows.

**Example 1.**

![Example Diagram]

From the above graphs we observe that \( \gamma^N_{2,2}(C_5) \leq 1 \neq 3 = n - 2, \gamma^N_{2,2}(C_6) \leq 2 \neq 4 = n - 2 \) and \( \gamma^N_{2,2}(C_7) \leq 3 \neq 5 = n - 2 \). From Lemma 3 and Example 1, we can have following result.

**Remark 1.** For \( n \geq 8 \), \( \gamma^N_{2,2}(C_n) \leq [n/5] + 3 < n - 2 \).

**Lemma 4.** Let \( n \geq 5 \) be an integer. Then the path \( P_n \) admits 2-DNNSDF with
The graphs $\gamma_{2,5}(P_5)$ is given by $V(P_5) \rightarrow \{-1,+1\}$ define by $f(a_i) = -1$, when $i = 5l$ or $i = 5l - 4, 1 \leq l \leq k$ and other wise $f(a_i) = +1$.

Proof. Let $V(P_n) = \{a_i/1 \leq i \leq n\}$ and $E(P_n) = \{a_i a_{i+1}/1 \leq i \leq n - 1\}.

Case 1: Suppose $n = 5k$ or $n = 5k + 3$ or $n = 5k + 4$ for $k \geq 1$. A 2-DNNSDF $f$ on $P_n$ is given by $V(P_n) \rightarrow \{-1,+1\}$. Define $f(a_i) = -1$, when $i = 5l$ or $i = 5l - 4, 1 \leq l \leq k$ and otherwise $f(a_i) = +1$.

Case 2: Suppose $n = 5k + 1$ or $5k + 2$ for $k \geq 1$. A 2-DNNSDF $f$ on $P_n$ is given by $V(P_n) \rightarrow \{-1,+1\}$ define by $f(a_i) = -1$, when $i = 5l$ or $i = 5l - 4, 1 \leq l \leq k$.

From the above labeling, it is easy to observe that at least 3 vertices of $N_3[a_i]$ must have +1 sign and hence $f(N_3[a_i]) \geq 1$ for all $i, 3 \leq i \leq n - 2$. Also the first and last four vertices have at least two vertices of +1 sign. Hence $f(N_3[a_i]) \geq 0$ when $i = 2, n - 2$. Also the first and last three vertices have at least two vertices of +1 sign. Hence $f(N_3[a_i]) \geq 1$ when $i = 1, n$.

Thus from the above labeling the result follows.

Example 2.

![Graph](https://via.placeholder.com/150)

From the above graphs we observe that $\gamma_{2,5}(P_5) \leq 1 \neq 3 = n - 2$. Therefore $\gamma_{2,5}(P_5) = 2 \neq 4 = n - 2$.

From Lemma 3 and Example 2, we can have following result.

Remark 2. For $n \geq 7$, $\gamma_{2,5}(P_n) \leq \lfloor n/5 \rfloor + 32 < n - 2$.

Lemma 5. Let $G$ be a connected graph of order $n$. Then $\gamma_{2,5}(G) = n - 2$ if and only if $G \cong P_2, P_3$ or $C_3$.

Proof. Let $\gamma_{2,5}(G) = n - 2$. We claim that $\Delta(G) \leq 2$. Assume, to the contrary, that $\Delta(G) \geq 3$. Let $v$ be a vertex of maximum degree and let $N_2(v) = \{v_1, \ldots, v_\lambda\}$. If $N_2[v_i] \cap N_2[v_j] = \{v\}$ for some $i = j$, then define $f : V(G) \rightarrow \{-1,+1\}$ by $f(v_i) = f(v_j) = -1$ and $f(x) = 1$ for all other vertices $x$. Clearly, $f$ is a 2-DNNSDF of $G$ with weight $n - 4$ which leads to a contradiction. Assume that $N_2[v_i] \cap N_2[v_j] = \{v\}$ for every pair $i,j, 1 \leq i = j \leq \lambda$. It is easy to see that the function $f : V(G) \rightarrow \{-1,+1\}$ defined by $f(v) = f(v_1) = -1$ and $f(x) = 1$ for all other vertices $x$, is a 2-DNNSDF of $G$ of weight $n - 4$ which leads to a contradiction. Therefore $\Delta(G) \leq 2$ and so $G$ is a path or cycle. By Remark 1 and 2, that is not possible to $\gamma_{2,5}(G) = n - 2$.

![Graph](https://via.placeholder.com/150)

Note that for the graphs $C_4$ and $P_4$, we have $\gamma_{2,5}(C_4) = \gamma_{2,5}(P_4) = 0 \neq n - 2$. Therefore $P_2, P_3$ and $C_3$ are the only graphs in which $\gamma_{2,5}(G) = n - 2$. The graphs $P_2, P_3$ and $C_3$ admit $k$-DNNSDF with $\gamma_{2,5}(P_2) = 0$, $\gamma_{2,5}(P_3) = 1$ and $\gamma_{2,5}(C_3) = 1$.

Lemma 6. Let $n \geq 5$ be an integer. Then the graph $C_n^+$ admits 2-DNNSDF with $\gamma_{2,5}(C_n^+) \leq 0$. 

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Proof. Let $V(C_n^+) = \{a_i, b_i / 1 \leq i \leq n\}$ and $E(C_n^+) = \{a_i, a_{i+1} / 1 \leq i \leq n - 1\} \cup \{a_1, a_n\} \cup \{a_i, b_i / 1 \leq i \leq n\}$. Define a function $f : V(C_n^+) \to \{-1, +1\}$, $f(a_i) = +1$ and $f(b_i) = -1$ for $1 \leq i \leq n$. Now we consider the vertices $a_i$, $N_2[a_i] = \{a_{i-2}, a_{i-1}, a_i, a_{i+1}, a_{i+2}, b_{i-1}, b_i, b_{i+1}\}$, by the above labeling $f(N_2[a_i]) = 2$ for $1 \leq i \leq n$. Next, we consider the vertices $b_i$, $N_2[b_i] = \{b_i, a_{i-1}, a_i, a_{i+1}\}$, by the above labeling $f(N_2[b_i]) = 2$ for $1 \leq i \leq n$. Thus $f$ is 2-DNNSDF with $\gamma_{2, s}(C_n^+) \leq 1$.

**Theorem 2.** Let $n \geq 5$ be an integer. Then the graph $C_n$ admits 2-DNNSDF with $\gamma_{2, s}(C_n) \leq 0$ when $n$ is even and $\gamma_{2, s}(C_n) \leq 1$ when $n$ is odd.

Proof. Let $V(C_n^-) = \{a_i / 1 \leq i \leq n\}$. Define a function $f : V(C_n^-) \to \{-1, +1\}$ by $f(a_i) = +1$ when $n$ is odd and $f(a_i) = -1$ when $n$ is even for $1 \leq i \leq n$. Note that $N_2[a_i] = V(C_n^-)$ for $1 \leq i \leq n$. Suppose $n$ is odd, then by the above labeling $f(N_2[a_i]) = \frac{n+1}{2} + \frac{n-1}{2} = 1$. Thus $f$ is 2-DNNSDF with $\gamma_{2, s}(C_n^-) \leq 1$. Suppose $n$ is even, then by the above labeling $f(N_2[a_i]) = \frac{n+1}{2} + \frac{n-1}{2} = 1$. Thus $f$ is 2-DNNSDF with $\gamma_{2, s}(C_n^-) \leq 0$.

After studying the above results, we find the following more general result:

**Theorem 3.** If $\text{diam}(G) \geq k$, then $G$ admits $k$-DNNSDF.

Proof. Since $\text{diam}(G) \geq k$, for every vertex $v \in V(G)$, we have $N_k[v] = V(G)$. Suppose $n = 2p$. Then we can label $p$ vertices with +1 signs and $p$ vertices with -1 signs. In this case, $f(N_k[v]) = p(1) + p(-1) = 0$.

Suppose $n = 2p + 1$. Then we can label $p + 1$ vertices with +1 signs and $p$ vertices with -1 signs. In this case, $f(N_k[v]) = (p + 1)(1) + p(-1) = 1$. Thus $G$ admits $k$-DNNSDF.

The next result follows immediately from the above theorem.

**Lemma 7.** The complete graph $K_n$ admits 2-DNNSDF for $n \geq 1$.

For the integers $m, n \geq 1$, the complete bipartite graph $K_{m,n}$ admits 2 DNNSDF.

The wheel graph $W_n$ admits 2-DNNSDF for $n \geq 3$.

The graph $G = P_m + P_n$ admits 2-DNNSDF for $m, n \geq 1$.

The friendship graph $T_n$ admits 2-DNNSDF.

**REFERENCES**
