RESTRAINED GIRTH DOMINATION NUMBER OF GRAPHS

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Abstract: The concept of complete graphs with real life application was introduced in [18] and the Forbidden pairs and the existence of a dominating cycle was introduced in[19]. In this paper, we introduce a new domination parameter called girth 2-domination number, that is, if all the edges of the girth(cycle) graph are the edges of any other cycles in a graph G and let G is a connected graph then $C_{ni}$ is the girth graph of G if $C_{ni} \leq \gamma$ otherwise stated the graph G has n vertices and m edges. Degree of a vertex v is denoted by d(v), the maximum degree of a graph G is denoted by $\Delta(G)$. Let $C_{i}$ a cycle on n vertices, $P_{n}$ a path on n vertices by and a complete graph on n vertices by $K_{n}$. A graph is connected if any two vertices are connected by a path. A maximal connected subgraph of a graph G is called a component of G. The number of components of G is denoted by $\omega(G)$. The complement $G'$ of G is the graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. A tree is a connected acyclic graph. A bipartite graph is a graph whose vertex set can be divided into two disjoint sets $V_1$ and another in $V_2$. A complete bipartite graph is a bipartite graph with partitions of order $|V_1|=m$ and $|V_2|=n$, denoted by $K_{m,n}$. A star denoted by $K_{1,n}$ is a tree with one root vertex and n-1 pendant vertices. A bistar, denoted by $B(m,n)$ is the graph obtained by joining the root vertices of the stars denoted by $F_{m}$ can be constructed by identifying n copies of the cycle $C_{m}$ at a common vertex. A wheel graph denoted by $W_{n}$ is a graph with n vertices formed by connecting a single vertex to all vertices of $C_{n}$. A Helm graph denoted by $H_{n}$ is a graph obtained from the wheel $W_{n}$ by attaching a pendant vertex to each vertex in the outer cycle of $W_{n}$.

The chromatic number of a graph G denoted by $\chi(G)$ is the smallest number of colors needed to colour all the vertices of a graph G in which adjacent vertices receive different colours. For any real number x, $[x]$ denotes the largest integer greater than or equal to x and $\lfloor x \rfloor$ the smallest integer less than or equal to x. A Nordhaus-Gaddum -- type result is a lower or upper bound on the sum or product of a parameter of a graph and its complement. Throughout

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I Introduction:

The concept of domination in graphs evolved from a chess board problem known as the Queen problem to find the minimum number of queens needed on an 8x8 chess board such that each square is either occupied or attacked by a queen. C.Berge[3] in 1958 and 1962 and O.Ore[8] in 1962 started the formal study on the theory of dominating sets. Thereafter several studies have been dedicated in obtaining variations of the concept. The authors in [7] listed over 1200 papers related to domination in graphs in over 75 variation.

Throughout the paper, G(V,E) a finite, simple, connected and undirected graph where V denotes its vertex set and E its edge set. Unless otherwise stated the graph G has n vertices and m edges. Degree of a vertex v is denoted by $d(v)$, the maximum degree of a graph G is denoted by $\Delta(G)$. Let $C_{i}$ a cycle on n vertices, $P_{n}$ a path on n vertices by and a complete graph on n vertices by $K_{n}$. A graph is connected if any two vertices are connected by a path. A maximal connected subgraph of a graph G is called a component of G. The number of components of G is denoted by $\omega(G)$. The complement $G'$ of G is the graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. A tree is a connected acyclic graph. A bipartite graph is a graph whose vertex set can be divided into two disjoint sets $V_1$ and another in $V_2$. A complete bipartite graph is a bipartite graph with partitions of order $|V_1|=m$ and $|V_2|=n$, denoted by $K_{m,n}$. A star denoted by $K_{1,n}$ is a tree with one root vertex and n-1 pendant vertices. A bistar, denoted by $B(m,n)$ is the graph obtained by joining the root vertices of the stars denoted by $F_{m}$ can be constructed by identifying n copies of the cycle $C_{m}$ at a common vertex. A wheel graph denoted by $W_{n}$ is a graph with n vertices formed by connecting a single vertex to all vertices of $C_{n}$. A Helm graph denoted by $H_{n}$ is a graph obtained from the wheel $W_{n}$ by attaching a pendant vertex to each vertex in the outer cycle of $W_{n}$.

The chromatic number of a graph G denoted by $\chi(G)$ is the smallest number of colors needed to colour all the vertices of a graph G in which adjacent vertices receive different colours. For any real number x, $[x]$ denotes the largest integer greater than or equal to x and $\lfloor x \rfloor$ the smallest integer less than or equal to x. A Nordhaus-Gaddum -- type result is a lower or upper bound on the sum or product of a parameter of a graph and its complement. Throughout

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In this paper, we only consider undirected graphs with no loops. The basic definitions and concepts used in this study are adopted from [11].

Given a graph \( G = (V(G), E(G)) \), the cardinality \( |V(G)| \) of the vertex set \( V(G) \) is the order of \( G \) is \( n \). The distance \( d_G(u,v) \) between two vertices \( u \) and \( v \) of \( G \) is the length of the shortest path joining \( u \) and \( v \). If \( d_G(u,v) = 1 \), \( u \) and \( v \) are said to be adjacent.

For a given vertex \( v \) of a graph \( G \), the open neighbourhood of \( v \) in \( G \) is the set \( N_G(v) \) of all vertices of \( G \) that are adjacent to \( v \).

The degree \( \deg_G(v) \) of \( v \) refers to \( |N_G(v)| \), and \( \Delta(G) = \max\{|\deg_G(v)| : v \in V(G)\} \). The closed neighbourhood of \( v \) is the set \( N_G[v] = N_G(v) \cup \{v\} \) for \( S \subseteq V(G) \). The minimum cardinality among dominating sets in \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \).

Definition [17]: In a connected graph \( G \), a chord of a spanning tree \( T \) is a line of \( G \) which is not in \( T \). Clearly the subgraph of \( G \) consists of \( T \) and any chord of \( T \) has exactly one cycle.

Definition [17]: If \( T \) is a regular of degree 2, every component is a cycle and regular graphs of degree 3 are called cubic.

Definition [17]: If all the edges of the girth are the edges of any other cycles in a graph \( G \).

Theorem [17]: Let \( x \) be a line of a connected graph \( G \). The following statements are equivalent:
1. \( x \) is a bridge of \( G \).
2. \( x \) is not on any cycle of \( G \).
3. There exist points \( u \) and \( v \) of \( G \) s.t the line \( x \) is on every path joining \( u \) and \( v \).
4. There exists a partition of \( v \) into subsets \( U \) and \( W \) s.t for any points \( u \in U \) and \( w \in W \) the line \( x \) is on every path joining \( u \) and \( w \).

Theorem [17]: Let \( G \) be a connected graph with at least three points. The following statements are equivalent:
1. \( G \) is a block.
2. Every two points of \( G \) lie on a common cycle.
3. Every point and line of \( G \) lie on a common cycle.
4. Every two lines of \( G \) lie on a common cycle.
5. Given two points and one line of \( G \), there is a path joining the points which contains the line.
6. For every three distinct points of \( G \), there is a path joining any two of them which contains the third.

A set \( S \subseteq V(G) \) is called a girth dominating set of \( G \) if every vertex in \( V-S \) is adjacent to at least one vertex in the girth cycle of \( G \). The minimum cardinality of a girth dominating set of \( G \) is called its girth domination number of \( G \) denoted by \( \gamma_g(G) \) [19].

II. MAIN RESULT

2.0. Definition: A subset \( S \) of \( V \) of a nontrivial graph \( G \) is said to be restrained girth dominating set, if every vertex in \( V-S \) is adjacent to at least one vertex in the girth cycle of \( G \). The minimum cardinality taken over all restrained girth dominating set is called the restrained girth domination number and is denoted by \( \gamma_{rg}(G) \).

2.1. Example: For any graph \( |G| = C_4 + v = 5 \) (or) \( |G| = C_3 + e = 5 \). If \( N(v_i) = (u_i, v_i) \) and \( V(G) = (C_3, v_i) \) where \( u_i \in S \) and \( v_i \in V-S \) has a restrained girth dominating set of \( G \) with \( \gamma_{rg}(G) = n-2=3 \) for \( n = 5 \) if \( \max\{|d(u_i, u_j)|\} \geq 3 \) where \( i \neq j \) and \( u_i \in C_3 \) by fig:1

Fig1: \( |G| = C_4 + v = 5, \gamma_{rg}(G) = n-2=3 \)
2.2. Result: For any graph $|G| = K_n = C_{n-2} + e = n$, $C_{n-2} = n - e$ and $\cup N(v_i) = (C_{n-2}, v_i)$ has a girth dominating set of $G$ with $\gamma_{rg}(G) \geq n - 2$ for $n \geq 5$ for every $v_i \in V - S$. Since if $\max \{d(u_i, u_j)\} = 2$ then we can have the restrained girth dominating set and its $\gamma_{rg}(G) = 3$. If $\max \{d(u_i, u_j)\} = 3$ then we can have $\gamma_{rg}(G) = 4$. Similarly we can have the restrained girth dominating set with $\gamma_{rg}(G) = k$ if $\max \{d(u_i, u_j)\} = k - 1$ and $|\cup N(v_i)| \geq k + 2$ for every $v_i \in V - S$.

2.3. Example: For any graph $|G| = K_n \circ K_{1, n-1} = [C_{n-1} + v] \circ K_{1, n-1} = 2n - 1$, $[C_{n-1}] \circ K_{1, n-1} + v = 2n - 1$ and $\cup N(v_i) = C_{n-1}$ has a girth dominating set of $G$ but it is not a restrained girth dominating set of $G$ since $\cup N(v_i) \neq (C_{n-1}, v_i)$ for $n \geq 4$ by fig:2.

![Fig 2: $|G| = K_n \circ K_{1, n-1}$, Not a restrained girth dominating set](image)

**Fig 2:** $|G| = K_n \circ K_{1, n-1}$, Not a restrained girth dominating set

2.4. Result: For any graph $|G| = K_n - 4e = (C_{n-3} + 3v) - 4e = K_6 - 4e = n$, $(C_{n-3} + 3v) = n$ and $C_{n-3} = n - 3$ with $\cup N(v_i) = (C_{n-3}, v_i)$ has a restrained girth dominating set of $G$ with $\gamma_{rg}(G) \geq n - 3$ for $n \geq 6$ for every $v_i \in V - S$ by fig:3. Since if $\max \{d(u_i, u_j)\} = 2$ then we can have the restrained girth dominating set and its $\gamma_{rg}(G) = 3$. If $\max \{d(u_i, u_j)\} = 3$ then we can have $\gamma_{rg}(G) = 4$. Similarly we can have the restrained girth dominating set with $\gamma_{rg}(G) = k$ if $\max \{d(u_i, u_j)\} = k - 1$ and $|\cup N(v_i)| \geq k + 3$ for every $v_i \in V - S$. By fig:3.

![Fig 3: $G = K_n - 4e = (C_{n-3} + 3v) - 4e$, $\gamma_{rg}(G) = 3$, for $n \geq 6$](image)

**Fig 3:** $G = K_n - 4e = (C_{n-3} + 3v) - 4e$, $\gamma_{rg}(G) = 3$, for $n \geq 6$

2.5. Result: For any graph $|G| = K_n - 6e = (C_{n-3} + 3v) - 6e = K_6 - 6e = n$, $(C_{n-3} + 3v) - 6e = n$ and $C_{n-3} = n - 3$ with $\cup N(v_i) = (C_{n-3}, v_i)$ has a restrained girth dominating set of $G$ with $\gamma_{rg}(G) \geq n - 3$ for $n \geq 6$ for every $v_i \in V - S$ by fig:4. Since if $\max \{d(u_i, u_j)\} = 2$ then we can have the restrained girth dominating set and its $\gamma_{rg}(G) = 3$. If $\max \{d(u_i, u_j)\} = 3$ then we can have $\gamma_{rg}(G) = 4$. Similarly we can have the restrained girth dominating set with $\gamma_{rg}(G) = k$ if $\max \{d(u_i, u_j)\} = k - 1$ and $|\cup N(v_i)| \geq k + 3$ for every $v_i \in V - S$ by fig:4.

![Fig 4: $G = K_n - 6e = (C_{n-3} + 3v) - 6e$, $\gamma_{rg}(G) = 3$, for $n \geq 6$](image)

**Fig 4:** $G = K_n - 6e = (C_{n-3} + 3v) - 6e$, $\gamma_{rg}(G) = 3$, for $n \geq 6$
2.6. Result: For any graph \(|G|=K_n - 5e = |(C_{n-3} + 3v) - 5e|=K_6 - 5e = n\), \((C_{n-3} + 3v) - 5e = n\) and \(C_{n-3} = n - 3\) with \(\cup N(v_i) = (C_{n-3}, v_i)\) has a restrained girth dominating set of \(G\) with \(\gamma_{rg}(G) \geq n-3\) for \(n \geq 6\) for every \(v_i \in V - S\). Since if \(\text{Max}\{d(u_i, u_j)\} = 2\) then we can have the restrained girth dominating set and its \(\gamma_{rg}(G)=3\). If \(\text{Max}\{d(u_i, u_j)\} = 3\) then we can have \(\gamma_{rg}(G)=4\). Similarly we can have the restrained girth dominating set with \(\gamma_{rg}(G)=k\) if \(\text{Max}\{d(u_i, u_j)\} = k-1\) and \(|\cup N(v_i)| \geq k + 3\) for every \(v_i \in V - S\).

2.7. Example: For any wheel graph \(G=W_n\), \(n=4\) is a restrained girth dominating set with \(\gamma_{rg}(G)=3\) with \(|\cup N(v_i)| = 4\), for every \(v_i \in V - S\) by fig:5.

2.8. Example: For any complete bipartite graph \(G=K_{m,n}\) is a restrained girth dominating set and its restrained girth domination number \(\gamma_{rg}(G)=4\) for \(m, n \geq 3\) since \(\text{Max}\{d(u_i, u_j)\} = 3\) and \(\text{Min}\{d(u_i, u_j)\} = 1\) and \(|\cup N(v_i)| = 6\), for every \(v_i \in V - S\) by fig:6.

2.9. Example: For any Helm graph \(G=H_n\) is not a girth dominating set for \(n \geq 4\). Since \(|N(u_i) \cap (V - S)| \neq 1\), \(i \neq 1\) where \(V - S = (H_n - C_3)\). Hence it is not a restrained girth dominating set of \(G\).

2.10. Example: For any graph \(G=K_4 - e\) is a girth dominating set and its girth domination number \(\gamma_{rg}(G)=3\) with \(|M|=1\) then we have \(|N(u_i) \cap (V - S)| = 1\), \(i \neq 1\) but \(\cup N(v_i) \neq (C_{n-1}, v_i) \neq (S, V - S)\) by fig:7. Hence it is not a restrained girth dominating set of \(G\). By fig.7.

2.11. Example: For any graph \(G=K_5 - ie\); \(i=1, 2, 3\) is a restrained girth dominating set and its restrained girth domination number \(\gamma_{rg}(G)=3\) by fig:8.

Fig 5: \(G=W_n\), \(n=4\), \(\gamma_{rg}(G)=3\).

Fig 6: \(G=K_{m,n}\); \(m, n \geq 3\), \(\gamma_{rg}(G)=4\).

Fig 7: \(G=K_4 - e\), \(\gamma_{rg}(G)=3\).

Fig 8: \(G=K_5 - 2e\), \(\gamma_{rg}(G)=3\).
2.12. Example: For any graph \( G = K_5 - 4e \) is a restrained girth dominating set and its restrained girth domination number \( \gamma_{rg}(G) = 3 \) with \(|M| = 1\) then we have \( |N(u_i) \cap (V - S)| = 1 \), \( i \neq j \) by fig.9.

![Image](image-url)

**Fig 9:** \( K_5 - 4e, \gamma_{rg}(G) = 3 \)

2.13. Lemma: Let \( G \) is a connected graph then \( C_{ni} \) is the girth graph of \( G \) if \( C_{ni} \leq C_{nj} \), \( i \neq j \).

2.14. Lemma: Let \( G \) be any graph and \( C_{ni} \) is the cycle then \( \cap C_{ni} \geq e \) if \( C_{ni} \leq C_{nj} \), \( i \neq j \).

**Proof:** Given \( G \) be a connected graph , By the definition of a cycle , \( \exists C_{ni} \leq C_{nj} \) and \( i \neq j \).

We have atleast one edge is common to \( C_{ni} \leq C_{nj} \). Hence we have \( \cap C_{ni} \geq e \).

2.15. Lemma: Let \( G \) be any connected graph and \( C_{ni} \) is the cycle then \( \cap C_{ni} \geq 2v \) if \( C_{ni} \leq C_{nj} \), \( i \neq j \).

**Proof:** Given \( G \) be a connected graph , By the definition of a cycle , \( \exists C_{ni} \leq C_{nj} \) and \( i \neq j \).

We have at least two vertices are in common to \( C_{ni} \leq C_{nj} \). Hence we have \( \cap C_{ni} \geq 2v \).

2.16. Lemma: Let \( G \) be a connected graph and \( \exists u_i \in C_{ni} \) is the restrained girth dominated if \( U^1_{i=1} N(v_i) = (u_i, v_i) = (S, V - S) = V \) Where \( v_i \in V-S \).

**Proof:** Given \( G \) be a connected graph , By lemma 2.13, \( \exists C_{ni} \leq C_{nj} \) and \( i \neq j \) and given it is a restrained girth dominated which gives \( |N(u_i) \cap (V - S)| = 1 \) and \( N(v_i) = u_i \) Where \( u_i \in C_{ni} \) and \( v_i \in V-S \), Hence \( U^1_{i=1} N(v_i) = (u_i, v_i) = (S, V - S) = V \).

2.17. Lemma: For any graph \( |G| = C_{n-2} + e = n \) then \( S = C_{n-2} = n-e \) where \( C_{n-2} \) is the restrained girth dominating set of a graph \( G \) and \( v \in V - S \) if then \( |N(u_i) \cap (V - S)| = 1 \) and \( U^1_{i=1} N(v_i) = (u_i, v_i) = (S, V - S) = V \).

**Proof:** For any graph \( |G| = n \) and if there exists a cycle \( C_{n-2} \leq C_{n-1} \) that is \( C_{n-2} + e = n \) we have \( S = C_{n-2} = n - 2v \) means that the graph is \( G-2v \) and the vertex \( v \) is non adjacent with any vertex of \( G \). If \( v_i \) is adjacent with atleast one vertex. Hence \( G = C_{n-2} + e \) then \( |N(u_i) \cap (V - S)| = 1 \) and \( U^1_{i=1} N(v_i) = (u_i, v_i) = (S, V - S) = V \) where \( v_i \in V-S \).

2.18. Lemma: For any graph \( |G| = C_{n-2} + e = n \) then \( S = C_{n-2} = n-2 \) where \( C_{n-2} \) is the restrained girth dominating set of a graph \( G \) and \( v \in V - S \) if \( |N(u_i) \cap (V - S)| = 2 \) with \( U^1_{i=1} N(v_i) = (u_i, v_i) = (S, V - S) = V \) where \( v_i \in V-S \).

**Proof:** For any graph \( |G| = n \) and if there exists a cycle \( C_{n-2} \leq C_{n-1} \) that is \( C_{n-2} + e = n \) where \( v \in V - S \) we have \( S = C_{n-2} = n - 2v \) means that the graph is \( G-e \). There fore the vertices \( v_i \geq 2 \) is non adjacent with any two vertices of \( G \). If \( v_i \) is adjacent with atleast 2 vertices the \( |N(u_i) \cap (V - S)| = 2 \) and \( U^1_{i=1} N(v_i) = (u_i, v_i) = (S, V - S) = V \) where \( v_i \in V-S \).

2.19. Lemma: Let \( G \) be any complete graph and \( \exists u_i \in C_{ni} \leq S \) is the girth dominating set of \( G \) then \( |N(u_i) \cap (V - S)| \geq 1 \) where \( v_i \in V-S \) with \( U^1_{i=1} N(v_i) = (u_i, v_i) = (S, V - S) = V \).

**Proof:** Given \( G \) be a connected graph by lemma 2.13, \( \exists C_{ni} \leq C_{nj} \) and \( i \neq j \) and given it is a restrained girth dominated ,which gives \( U^1_{i=1} N(v_i) = (u_i, v_i) \) where \( u_i \in C_{ni} \), \( i = 1, 2 \ldots n \) and then for any graph \( |G| = n \) and if there exists a cycle \( C_{n-2} \leq C_{n-1} \) that is \( C_{n-2} + e = n \) we have \( S = C_{n-2} = n - 2v \) means that the graph is \( G-e \) and the vertex \( v \) is non adjacent with any vertex of \( G \). If \( v_i \) is adjacent with atleast one vertex then \( |N(u_i) \cap (V - S)| \geq 1 \) where \( v_i \in V-S \) and \( U^1_{i=1} N(v_i) = (u_i, v_i) = (S, V - S) = V \).
2.20. **Theorem**: For any graph $G$. Let $S$ be a restrained girth dominating set of $G$ if $S=\{u_i\}$ for $i=1,2,...n$. is $\gamma_{rg}(G-v) \leq \gamma_{rg}(G)$

**Proof**: For any graph $G$. Let $S$ be a restrained girth dominating set of $G$ and $V-S=\{v_i\}$ for $i=1,2,...,n$. we have given $G$ be a connected graph by lemma 2.13, $\exists C_{ni} \leq C_{nj}$ and $i \neq j$ and given it is restrained girth dominated, which gives $U_{i=1}^n N(v_i)=(u_i,v_i)$ where $u_i \in C_{ni}$, i.e., 1,2..n and then $|N(u_i) \cap (V-S)| \geq 1$ where $v_i \in V-S$.

If any one of $v_i$ is removed then the edges incident on $v_i$ is removed then it may not be the restrained girth dominating set of $G$. Since $U N(v_i) = (S,V-S)$ which gives $N(v_i) \neq \{u_i,v_i\}$. Therefore $U N(v_i) \neq (S,V-S) = V$. Hence $\gamma_{rg}(G-v) \leq \gamma_{rg}(G)$.

2.21. **Theorem**: For any graph $G$ with girth cycle is girth dominated if it has at least one matching.

**Proof**: For any graph with girth cycle will be the restrained girth dominating set $S$ of $G$. If every vertex of $V-S$ is adjacent to at least one vertex of $S$ also another vertex of $V-S$. If not then there exist one $v \in V-S$ is not adjacent to one vertex of $S$ then we made a matching to this vertex with $S$ then it gives the restrained girth dominating set of $G$ with $|M|=1$ if $|N(u_i) \cap (V-S)| \geq 1$ and $U_{i=1}^n N(v_i)=(u_i,v_i) = (S,V-S) = V$. Hence we can find at least one matching to find the restrained girth dominating set of $G$, that is $|M| \geq 1$.

2.22. **Theorem**: Every restrained girth dominating set and it is of chromatic number $\chi(G) \geq 3$.

**Proof**: Given the graph $G$ is a restrained girth dominating set since it is restrained girth dominated the graph $G$ must have at least a girth cycle of $C_3$. If every cycle $C_n$ since if $\max\{d(u_i,u_j)\} = 2$ then we can have the restrained girth dominating set and its $\gamma_{rg}(G)=3$ must have 3 colourable and every vertex of $V-S$ is adjacent to at least one vertex of $S$ and also another vertex of $V-S$ have $4^{th}$ vertex may have the $3^{rd}$ colour other than the colours which had in the restrained girth cycle and which is adjacent to the vertex. Hence there must have at least 3 colours needed to colour the restrained girth dominating graph, that is $\chi(G) \geq 3$.

2.23. **Lemma**: If $|N(u_i) \cap (V-S)| \geq 1$. Then $(u_1,u_2,...,u_{n-1})$ are the restrained girth dominating set of any graph $G$.

**Proof**: For any graph $G=\{v\}$ and if there exists a cycle $C_{n-i} \leq C_n$, $|C_{n-i} + U_{i=1}^{n-3} v_i| = n$ where $v_i \in V-S$ and we have $S=C_{n-i} = n \cdot v_i$; $S=\{n-3\}$, $i < n-2$ means that the graph $G-v_i$. There fore any $v_i$ is non adjacent with any vertex of $G$. If $v_i$ is adjacent with $C_{n-i}$ by at least one vertex then $|N(u_i) \cap (V-S)| \geq 1$ then $U N(v_i) = (u_1,u_2,...,u_{n-1}) = V$ that is $\gamma_{rg}(v_i) = V$.

2.24. **Theorem**: Let $S$ be any restrained girth dominating set in a graph $G$ then their domination numbers are

(i) $\gamma_{rg}(G)=n-2 \leq 3$ if $|G|=C_n + v = 5$ (or) $|G|=C_3 + e = 5$ (or) $|G|=K_n = C_{n+2} + e = n$

(ii) $\gamma_{rg}(G) \geq n-3$ if $|G|=K_n - 2e = (C_{n-3} + 3v) - 2e = K_n - 2e - n$ (or) $|G|=K_n - 4e = (C_{n-3} + 3v) - 4e$ for $n \geq 6$.

(iii) $\gamma_{rg}(G)=(n-1) \leq 3$ if $G=K_n$, $n=4$

(iv) $\gamma_{rg}(G)=4$ if $G=K_m,n$ for $m,n \geq 3$ if $G=K_n = (C_{n-3} + 3v) - 2e$ for $n \geq 6$.

(v) $\gamma_{rg}(G)=3$ if $G= K_{5-i}$ for $i=1,2,3$ (or) $G= (K_5 - 4e)$ with $|M|=1$.

**Proof**: Given $S$ be a restrained girth dominating set of any graph $G$. By lemma 2.13, for any graph $|G|=n$ and if there exists a cycle $C_{n-i} \leq C_n$, $|C_{n-i} + U_{i=1}^{n-3} v_i| = n$ where $v_i \in V-S$ and we have $S=C_{n-i} = n \cdot v_i$; $S=\{n-3\}$, $i \leq n-3$ means that the graph $G-v_i$. There fore any $v_i$ is non adjacent with any vertex of $G$. If $v_i$ is adjacent with $C_{n-i}$ by at least one vertex then $|N(u_i) \cap (V-S)| \geq 1$ then $U N(v_i) = (u_1,u_2,...,u_{n-1})$ that is, $U N(v_i) = V$ and By Lemma 2.16 Given $G$ be a connected graph. By Lemma 2.13, $\exists C_{ni} \leq C_{nj}$ and $i \neq j$ and given it is a restrained girth dominated which gives $|N(u_i) \cap (V-S)| = 1$ and $N(v_i) = u_i$. Where $u_i \in C_{ni}$ and $v_i \in V-S$. Hence $U_{i=1}^n N(v_i)=(u_i,v_i) = (S,V-S) = V$.
Case(i) For any graph $|G| = C_4 + v = 5$ (or) $|G| = C_5 + e = 5$. If $N(v_i) = (u_i, v_i)$ and $U \cap N(v_i) = (C_3, v_i)$ where

$u_i \in S$ and $v_i \in V - S$ has a restrained girth dominating set of $G$ with $\gamma_r(G) = n-2=3$ for $n=5$ if $\max\{d(u_i, u_j)\} \geq n-3, i \neq j$ where $u_i \in C_3$.

Subcase(a) Suppose for any graph $|G| = K_n = C_{n-2} + e = n$. $C_{n-2} = n - e$ and $U \cap N(v_i) = (C_{n-2}, v_i)$ has girth dominating set of $G$ with $\gamma_r(G) \geq n-2$ for $n \geq 5$ for every $v_i \in V - S$. Since if $\max\{d(u_i, u_j)\} = 2$ then we can have the restrained girth dominating set and its $\gamma_r(G) = 3$. If $\max\{d(u_i, u_j)\} = 3$ then we can have $\gamma_r(G) = 4$. Similarly we can have the restrained girth dominating set with $\gamma_r(G) = k$ if $\max\{d(u_i, u_j)\} = k-1$ and $|U \cap N(v_i)| \geq k + 2$, for every $v_i \in V - S$.

Case(ii) For any graph $|G| = K_n - 2e = (C_{n-3} + 3v) - 2e = K_6 - 2e = n$. $(C_{n-3} + 3v) = n$ and $C_{n-3} = n - 3$ with $U \cap N(v_i) = (C_{n-3}, v_i)$ has girth dominating set of $G$ with $\gamma_r(G) \geq n-3$ for $n \geq 6$ for every $v_i \in V - S$.

Since if $\max\{d(u_i, u_j)\} = 2$ then we can have the restrained girth dominating set and its $\gamma_r(G) = 3$. If $\max\{d(u_i, u_j)\} = 3$ then we can have $\gamma_r(G) = 4$.

Similarly we can have the restrained girth dominating set with $\gamma_r(G) = k$ if $\max\{d(u_i, u_j)\} = k-1$ and $|U \cap N(v_i)| \geq k + 3$, for every $v_i \in V - S$.

Subcase(a) Suppose for any graph $|G| = K_n - 4e = (C_{n-3} + 3v) - 2e = K_6 - 2e = n$. $(C_{n-3} + 3v) = n$ and $C_{n-3} = n - 3$ with $U \cap N(v_i) = (C_{n-3}, v_i)$ has a restrained girth dominating set of $G$ with $\gamma_r(G) \geq n-3$ for $n \geq 6$ for every $v_i \in V - S$.

Similarly we can have the restrained girth dominating set with $\gamma_r(G) = k$ if $\max\{d(u_i, u_j)\} = k-1$ and $|U \cap N(v_i)| \geq k + 3$, for every $v_i \in V - S$.

Case (iii) For any wheel graph $G = W_n$ where $n = 4$ is a restrained girth dominating set with $\gamma_r(G) = 3$ with $U \cap N(v_i) = (C_{n-1}, v_i)$ since $\max\{d(u_i, u_j)\} = 2$ then we can have the restrained girth dominating set and its $\gamma_r(G) = (n-1)=3$ and $|U \cap N(v_i)| = 4$ for every $v_i \in V - S$.

Case (iv) For any complete bipartite graph $G = K_{n,m}$ is a restrained girth dominating set and its restrained girth domination number $\gamma_r(G) = 4$ for $m, n \geq 3$ since $\max\{d(u_i, u_j)\} = 3$ and $\min\{d(u_i, u_j)\} = 1$ and $|U \cap N(v_i)| \geq 6$ for every $v_i \in V - S$.

Case (v) For any graph $G = K_5 - e$ where $i = 1, 2, 3$ is a restrained girth dominating set and its restrained girth domination number $\gamma_r(G) = 3$ with $U \cap N(v_i) = (C_3, v_i)$.

Subcase (a) For any graph $G = K_5 - 4e$ is a restrained girth dominating set and its restrained girth domination number $\gamma_r(G) = 3$ with $|M| = 1$ then we have $|N(u_i) \cap (V - S)| = 1, i \neq 1$ and $U \cap N(v_i) = (C_3, v_i)$.

2.25. Theorem: If $|U \cap N(v_i)| = (u_i, v_i)$ and $|U \cap N(v_i)| \geq 3$ then $u_i$ are the restrained girth dominating set of any graph $G$.

Proof: Given $G$ be any graph and we have in $|G| = |C_{n-k} + U_{i=1}^k v_i| = n$ then $S = C_{n-k} = n - U_{i=1}^k v_i$.

$N(U_{i=1}^k v_i) \cap (V - S) = n - k$. Hence $N(v_i)$ must have adjacent with $C_{n-k}$ then we have $|N(u_i) \cap (V - S)| = n - k$. Hence $N(v_i)$ must have adjacent with $C_{n-k}$ by at least one vertex also another vertex of $v_i$ gives $|N(u_i) \cap (V - S)| = 1$ and $N(v_i) = (u_i, v_i)$ and already we have $S \geq 3$. There fore we have $N(v_i) = (u_i, v_i)$ and $|U \cap N(v_i)| \geq 3$ and in general we have $N(v_i) = (u_1, u_2, ..., u_{n-i}, v_i)$, $|U \cap N(v_i)| = n - i$ where $u_i \in C_{n-i}, i = 1, 2, ..., n-1$ and $v_i \in V - S$. Hence $\gamma_r(G) \geq n-i$ and $\max\{d(u_i, u_j)\} \geq n-(i+1), i \neq j$.

2.26. Theorem: If $|U \cap N(v_i)| \geq k$ then $U_{i=1}^k u_i$ are the restrained girth dominating set of $G$. Then any two vertices of $C_n$ and $\max\{d(u_i, u_j)\} = k - 1, k \geq 3$ that is $i, j$ are non adjacent and $d(u_i, u_j) \neq 1$. 

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Proof: Given \( |\bigcup (v_i)| \geq k \) then \( U_{i=1}^k u_i \) are the restrained girth dominating set of \( G \) and we have \[ |G| = |C_n + U_{i=1}^k v_i|; n \geq r+k \text{ and } k \geq 3 \] we have \( S=n-r \) which implies that \( S \geq r+k-r \), that is \( S \geq k \) and if \( U_{i=1}^k v_i \) is non adjacent with any vertices of \( C_n \) then we have \( |N(u_i) \cap (V-S)| = n-r \).

Hence \( N(v_i) \) must have adjacent with \( C_{n-k} \) by at least one vertex that is \( |N(v_i) \cap (V-S)| = 1 \) and \( |\bigcup N(v_i)| \geq k \) and we have \( S \geq k \), \( \bigcup N(v_i) = (U_{i=1}^k u_i, v_i) \) and we must have \( d(u_i, u_j) = 1 \), \( i \neq j \) and nonadjacent if \( d(u_i, u_j) = 2 \) then we can have the restrained girth dominating set and \( \gamma_{rg}(G)=3 \).

If \( \max\{d(u_i, u_j)\} = 3 \) then we can have \( \gamma_{rg}(G)=4 \). Similarly we can have the restrained girth dominating set with \( \gamma_{rg}(G)=k \) if \( \max\{d(u_i, u_j)\} = k-1 \) and \( |\bigcup N(v_i)| \geq k \), for every \( v_i \in V-S \).

2.27. Theorem: Every connected graph is of restrained girth dominating set \( C_n \) with \( |M| \leq 3 \).

Proof: Let \( C_n \) be the girth subgraph of \( G \) and \( S = C_n \) with \( V-S=G-S \). If \( |\bigcup N(v_i)| = 3 \) and its \( \bigcup N(v_i) = (S, V-S) = V \) then \( C_n \) is the restrained girth dominating set of \( G \) with \( n \geq 3 \). If \( d(u_i, u_j) = 1 \) where \( u_i \in C_n \) then \( C_n \) is the girth subgraph of \( G \). If \( d(u_i, u_j) = 2 \) then add a chord to the subgraph which gives \( d(u_i, u_j) = 1 \) and \( \bigcup N(v_i) = 3 \) and \( \bigcup N(v_i) = (S, V-S) = V \) then it is the restrained girth dominating set of \( G \). If \( (V-S) = (C_3, V-S) \) and if \( |\bigcup N(v_i)| < 2 \) add one edge of matching with any one vertex of \( C_3 \) again if \( |\bigcup N(v_i)| < 2 \) add another vertex of \( C_3 \). Since the restrained girth graph is of cycle 3 we can add maximum 3 matchings then it becomes restrained girth dominating set of \( G \). That is \( |M| \leq 3 \). Otherwise the graph has no restrained girth dominating set.

2.28. Corollary: Every connected graph is of restrained girth dominating set \( C_n \) with \( |M| \leq n; \ n \geq 3 \).

Proof is obvious from Theorem 2.26 and 2.27.

2.29. Theorem: Every Corona graph of a restrained girth graph \( G \) is not a restrained girth dominating set, that is \( |\bigcup N(H_o)| \neq n-k \).

Proof: Let \( S \) be a restrained girth graph of any number of vertices of \( G \) and \( S \) be the set of \( n-k \) vertices of a restrained girth cycle then \( |\bigcup N(u_i) \cap N(H_o)| \neq n-k \) and \( V-S=G \cup H_o \). Hence we have \( N(H_o) \) of \( S, V-S \) also \( \bigcup N(H_o) \neq V \) which implies that \( |\bigcup N(H_o)| \neq (u_i, u_2, \ldots, u_{n-k}) = n-k \).

Hence for any graph \( |G| \geq n \) and if there exists a cycle \( C_{n-k} \leq C_{n-1} \) that is \( C_{n-k} + U_{i=1}^{n-3} v_i = n \), we have \( S \in C_{n-k} = n-i \), \( v_i = 2 \ldots k \) and \( k \geq 2 \) means that the graph is \( G \) and \( v \) is non adjacent with any vertex of \( G \) then it gives \( |\bigcup N(u_i) \cap (V-S)| = 1 \) and \( |\bigcup N(v_i)| \geq k \) and we have \( S \in C \), \( |\bigcup N(u_i) \cap (V-S)| = 1 \) and \( |\bigcup N(v_i)| \neq (u_i, u_2, \ldots, u_{n-k}) = n-k \).

Hence every vertex of \( V-S \) is non adjacent to at least one vertex of \( S \) that is \( |\bigcup N(H_o)| = (u_i, u_2, \ldots, u_{n-k}) = n-k \) and \( |\bigcup N(H_o) \cap N(H_o)| \neq n-k \).

Hence it is not a restrained girth dominating set of \( G \) by the definition restrained girth domination but if \( S \in G \) then \( |\bigcup N(u_i) \cap (V-S)| = 1 \) and \( \bigcup N(v_i) = (U_{i=1}^k u_i, v_i) \) which gives the restrained girth dominating set \( G \).

2.30. Lemma: Let \( G \) be any connected graph with a restrained girth dominating set then \( \bigcup N(S) = (S, V-S) \)

Proof: For any graph \( G \). Let \( S \) be a restrained girth dominating set of \( G \) and \( V-S = \{v_i\} \) for \( i=1, 2, \ldots, n \). we have given \( G \) be a connected graph by lemma 2.13, \( \exists C_{ni} \leq C_{nj} \) and \( i \neq j \) and given it is a restrained girth dominated which gives \( \bigcup N(v_i) = (U_{i=1}^k u_i, v_i) \) where \( u_i \in C_{ni} \), \( i=1, 2, \ldots, n \) and we have \( N(u_i) = (u_i, v_i) \) and \( \bigcup N(u_i) = (S, V-S) \) and \( |\bigcup N(u_i) \cap (V-S)| \geq 1 \) where \( v_i \in V-S \).

2.31. Theorem: Let \( S_1 \) and \( S_2 \) be any two restrained girth dominating set of same order in \( G_1 \) and \( G_2 \) respectively then \[ |\bigcup N(S_1) \cdot \bigcup N(S_2)| = |\bigcup N(S_1)| \cdot |\bigcup N(S_2)| = |S_1, V-S| \]

Proof: By lemma 2.30, For any graph \( G \). Let \( S \) be a restrained girth dominating set of \( G \) and \( V-S = \{v_i\} \) for \( i=1, 2, \ldots, n \). we have given \( G \) be a connected graph by lemma 2.13, \( \exists C_{ni} \leq C_{nj} \) and \( i \neq j \) and given it is a restrained girth dominated which gives \( \bigcup N(v_i) = (U_{i=1}^k u_i, v_i) \) where \( u_i \in C_{ni} \), \( i=1, 2, \ldots, n \) and we have \( N(u_i) = (u_i, v_i) \) and \( \bigcup N(u_i) = (S, V-S) \) and \( |\bigcup N(u_i) \cap (V-S)| \geq 1 \) where \( v_i \in V-S \).
Let $G_1$ and $G_2$ be any two graphs of $n_1$ and $n_2$ vertices and its restrained girth dominating sets are $S_1$ and $S_2$ respectively. If we have $\bigcup (S_1) = (S_1, V - S_1)$ and $\bigcup (S_2) = (S_2, V - S_2)$ and its $\bigcup (S_1) \cdot \bigcup (S_2) = [S_1 \cdot (V_1 - S_1) \cdot (V_2 - S_2)] = [S, V_1 \cdot V_2 - S]$ where $S_1, S_2 = S$ and $V_1, V_2 = V$ which gives $\bigcup (S) = [S, V - S]$.

III. Conclusion:

In this paper we found an upper bound for the Restrained girth domination number and relationship between Restrained girth domination numbers of graphs and characterized the corresponding extremal graphs. Similarly Restrained girth domination numbers with other graph theoretical parameters can be considered.

References

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