INTEGRO-DIFFERENTIAL EQUATIONS ON TIME SCALES (SOME APPROXIMATE SOLUTIONS)

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Abstract: This paper presents study some properties of new dynamic equations, integro-differential equations in two independent variables on time scales and some approximate solutions for certain integro-differential equations on time scales. The results were based on the explicit estimate of integral inequality on time scales.

Keywords: continuous dependence, explicit estimates, integral inequality, integro-differential equations, time scales

INTRODUCTION:

Time-scale calculus is a unification of the theory of difference equations with that of differential equations, unifying integral and differential calculus with the calculus of finite differences, offering formalism for studying hybrid systems. In the year 1998 German Mathematician Stefan Hilger has initiated the study of time scales in his Ph.D dissertation which unifies the continuous and discrete calculus. The study of dynamic equations on time scales has tremendous potential of application in various fields such as biological and Physical sciences where the discrete and continuous phenomena are simultaneously occurring. The method of finding the approximate solution is very powerful method in solving such applications.Many authors have studied various properties of dynamic equations on time scales [1,2,3]. Basic information about time scale calculus can be found in [4,5]. Many authors have studied various dynamic equations [6,7, 8,9,10] .An "integro-differential equation" involves both integrals and derivatives of an unknown function. By Using the Laplace transform of integrals and derivatives, an integro-differential equation can be solved. Many mathematical techniques have been developed to identify intrinsic timescales in dynamical systems.

Now, consider the approximate solution of the dynamic integro-differential equations on time scales of type

$$u^{\Delta}(t) = f\big(t, u(t), (Pu)(t), (Qu)(t)\big),\ u(0) = 0, \qquad (1.1)$$

With $(Pu)(t) = \int_{t_0}^{t} p\big(t, \tau, u(\tau)\big)\,\Delta\tau,\ \ (Qu)(t) = \int_{t_0}^{t} q\big(t, \tau, u(\tau)\big)\,\Delta\tau,$

Where f, p, q are given functions and u is unknown function to be found $f : I_T \times R^n \times R^n \to R^n$, $p, q : I_T \times I_T \times R^n \to R^n$, are continuous function, t is from time scale T which is nonempty.

Closed subset of R the set of real numbers $\tau \le t$ and $I_T = I \cap T, I = [t_0, \infty)$ the given subset of R. R^n is n dimensional Euclidean space. Let $u(t) \in C_{rd}(I_T, R_+)$ and satisfies the inequality

$$\left|u^{\Delta}(t) - f\left(t, u(t), (Pu)(t), (Qu)(t)\right)\right| \le \varepsilon,$$ For a given constant ε

Where $u(0) = u_0.$

MAIN DISCUSSION:
We now give the inequality which is used in proving out results.

Lemma 2.1

. Let $y, h, k, r \in C_{rd}(I_T, R_+)$ and

$$y(t) \le c + \int_{t_0}^{t} h(s)\left[y(s) + \int_{s_0}^{s} k(\tau)y(\tau)\Delta\tau + \int_{\xi_0}^{\xi} r(\tau)y(\tau)\Delta\tau\right]\Delta s, \quad (2.1)$$

for , $t \in I_T$ where $c \ge 0$ is constant, therefore

$$y(t) \le \frac{c}{1-\bar{r}} e_{(h+k)}(t, t_0), \quad (2.2)$$

Where

$$\bar{r} = \int_{\xi_0}^{\xi} r(\tau) e_{(h+k)}(\tau, \tau_0)\Delta\tau < 1. \quad (2.3)$$

Lemma 2.2. Suppose that the functions f, p, q in (1.1) satisfy the conditions

$$|f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w})| \le k(t)[|u-\bar{u}| + |v-\bar{v}| + |w-\bar{w}|], \quad (2.4)$$

$$\left|p(t,\tau,u) - p\left(t,\tau,\bar{u}\right)\right| \le l(t)g(\tau)\left|u-\bar{u}\right|, \quad (2.5)$$

$$|q(t,\tau,u) - q(t,\tau,\bar{u})| \le m(t)r(\tau)|u-\bar{u}|, \quad (2.6)$$

Where $k, l, g, m, r \in C_{rd}(I_T, R_+)$.
Let n(t)=max{k(t),k(t)l(t),k(t)m(t)} and $\bar{r}$ as in (2.3) for i=1,2. Let $u_i(t)$ be respectively

ε_i- approximate solution in equation (1.1) on I_T with $u_i(t_0) = \bar{u_\iota}$ such that

$$|\bar{u_1} - \bar{u_2}| \le \delta, \quad (2.7)$$

Where $\delta \ge 0$ is a constant

Therefore,

$$|u_1(t) - u_2(t)| \le \frac{\varepsilon t + \delta}{1-\bar{r}} e_{n+g}(t, t_0)\Delta t. \quad (2.8)$$

Proof. Since $u_i(t), t \in I_T$ are ε_i-approximate solution in equation (1.1) respectively with

$u_i(t_0) = \bar{u_\iota},$ we have

$$\left|u_i^{\Delta}(t) - f\big(t, u_i(t), (Pu_i)(t), (Qu_i)(t)\big)\right| \le \varepsilon_i, \tag{2.9}$$

for i=1,2. Delta integrating both sides of (2.7) from t_o to t over I_T we have

$$\varepsilon_i t \ge \int_{t_0}^{t} \left|u_i^{\Delta}(s) - f\big(s, u_i(s), (Pu_i)(s), (Qu_i)(s)\big)\right| \Delta s$$

$$\ge \left|\left\{\int_{t_0}^{t} u_i^{\Delta}(s) - f\big(s, u_i(s), (Pu_i)(s), (Qu_i)(s)\big)\right\}\Delta s\right|$$

$$= \left|\left\{u_1(t) - \overline{u_1} - \int_{t_0}^{t} f\big(s, u_i(s), (Pu_i)(s), (Qu_i)(s)\big)\right\}\right|, \tag{2.10}$$

For i=1,2.

From (2.10) and using the elementary inequalities

$$|z_1 - z_2| \le |z_1| + |z_2|, \quad |z_1| - |z_2| \le |z_1 - z_2|.$$

(2.11)

We have

$$(\varepsilon_1 + \varepsilon_2)t \ge \left|\left\{u_1(t) - \bar{u_1} - \int_{t_0}^{t} f\big(s, u_1(s), (Pu_1)(s), (Qu_1)(s)\big)\Delta s\right\}\right|$$

$$+ \left|\left\{u_2(t) - \bar{u_2} - \int_{t_0}^{t} f\big(s, u_2(s), (Pu_2)(s), (Qu_2)(s)\big)\Delta s\right\}\right|$$

$$\ge \left|\left\{u_1(t) - \bar{u_1} - \int_{t_0}^{t} f\big(s, u_1(s), (Pu_1)(s), (Qu_1)(s)\big)\Delta s\right\}\right.$$

$$\left. - \left\{u_2(t) - \bar{u_2} - \int_{t_0}^{t} f\big(s, u_2(s), (Pu_2)(s), (Qu_2)(s)\big)\Delta s\right\}\right|$$

$$\ge |u_1(t) - u_2(t)| - |\bar{u_1} - \bar{u_2}|$$

$$+ \left|\int_{t_0}^{t} f\big(s, u_1(s), (Pu_1)(s), (Qu_1)(s)\big)\Delta s - \int_{t_0}^{t} f\big(s, u_2(s), (Pu_2)(s), (Qu_2)(s)\big)\Delta s\right|. \tag{2.12}$$

Let $y(t) = |u_1(t) - u_2(t)|, t \in I_T$ and$\varepsilon_1 + \varepsilon_2 = \varepsilon$.

From (2.12) and using hypothesis we have

$$y(t) \le \varepsilon t + |\bar{u_1} - \bar{u_2}| + \int_{t_0}^{t} \Big|f\big(s, u_1(s), (Pu_1)(s), (Qu_1)(s)\big)$$

$$- \int_{t_0}^{t} f\big(s, u_2(s), (Pu_2)(s), (Qu_2)(s)\big)\Big| \Delta s$$

$$\le \varepsilon t + \delta + \int_{t_0}^{t} k(t)\left[y(s) + \int_{t_0}^{s} l(\tau)g(\tau)y(\tau)\Delta\tau + \int_{t_0}^{a} m(\tau)r(\tau)y(\tau)\Delta\tau\right]\Delta s$$

$$\leq \varepsilon t + \delta + \int_{t_0}^{t} n(t)\left[y(s) + \int_{t_0}^{s} g(\tau)y(\tau)\Delta\tau + \int_{t_0}^{a} r(\tau)y(\tau)\Delta\tau\right]\Delta s \qquad (2.13)$$

Observation 1. In case if $u_1(t)$ is a solution of equation (1.1) then we have $\varepsilon_1 = 0$and from (2.8) we see that $u_2(t) \to u_1(t)$ as $\varepsilon_2 \to 0$ and $\delta \to 0$. If we put$\varepsilon_1 = \varepsilon_2 = 0$, $\bar{u_1} = \bar{u_2}$ in (2.8) then we obtain the uniqueness of (1.1).

Now we consider the following dynamic integro-differential equation

$$w^{\Delta}(t) = \bar{f}\big(t, w(t), (Pw)(t), (Qw)(t)\big),\ \ w(0) = w_0, \qquad (2.14)$$

where P and Q are defined as in (1.1) and $\bar{f} \in C_{rd}(I_T \times R^n \times R^n \times R^n, R^n)$

Lemma 2.3. Suppose that f,p, q as in equation (1.1) satisfy the condition (2.4) to (2.6) and there exists $\bar{\varepsilon} \geq 0, \bar{\delta} \geq 0$ such that

$$\left|f(t, v_1, v_2, v_3) - \bar{f}(t, v_1, v_2, v_3)\right| \leq \bar{\varepsilon}, \qquad (2.15)$$

$$|u_0 - w_0| \leq \bar{\delta}, \qquad (2.16)$$

Where f, y_0 and $\bar{f}, w_0$ are as in equation (1.1) and (2.14). Let $h, \bar{r}$ be as in Theorem 2.2. Let u(t) and w(t) be respectively solution of equations (1.1) and (2.14) onI_T. Then

$$|u(t) - w(t)| \leq \frac{\bar{\varepsilon}+\bar{\delta}}{1-\bar{r}} e_{h+k}(t, t_0). \qquad (2.17)$$

Proof. Let$y(t) = |u(t) - w(t)|, t \in I_T$**.** Since u(t),w(t) are solutions of equation (1.1) & (2.14) we **h**ave $y(t) \leq |u_0 - w_0| + \int_{t_0}^{t}\left|f\big(s, u(s), (Pu)(s), (Qu)(s)\big)\right.$

$$-f\big(s, w(s), (Pw)(s), (Qw)(s)\big)\Big|\Delta s + \int_{t_0}^{t}\Big|f\big(s, w(s), (Pw)(s), (Qw)(s)\big)$$

$$-\bar{f}\big(s, w(s), (Pw)(s), (Qw)(s)\big)\Big|\Delta s \quad + \int_{t_0}^{t} s(t)\left[y(s) + \int_{t_0}^{s} l(t)g(\tau)y(\tau)\Delta\tau + \int_{t_0}^{a} m(\tau)r(\tau)y(\tau)\Delta\tau\right]\Delta s$$

$$\leq \left(\bar{\varepsilon}t + \bar{\delta}\right) + \int_{t_0}^{t} n(t)\left[y(s) + \int_{t_0}^{s} g(\tau)y(\tau)\Delta\tau + \int_{t_0}^{a} r(\tau)y(\tau)\Delta\tau\right]\Delta s. \qquad (2.18)$$

Observation 2. The result in Lemma 2.3 relates the solution of equation (1.1) and (2.14) in the sense that if f is close to $\bar{f}$ and if u_0 is close to w_0 the solution of equation (1.1) and (2.14) are close together.

APPROXIMATE SOLUTIONS:

Now we consider the following dynamic integro-differential equation

$$u^{\Delta}(t) = f_k\big(t, y(t), (Pu)(t), (Qu)(t)\big),\ \ u(0) = \alpha_k, \qquad (3.1)$$

for k=1,2,…, where P,Q are defined as in (1.1) and α_k is a sequence in R and

$f_k: I_T \times R^n \times R^n \to R^n$.

Lemma 3.1. Suppose that the functions f, g, h in equation (1.1) satisfy the conditions (2.4)-(2.6) and there exist constants $\varepsilon_k \geq 0, \delta_k \geq 0, (k = 1,2,\dots)$ such that

$$|f(t, u, v, w) - f_k(t, u, v, w)| \leq \varepsilon_k, \qquad (3.2)$$

$$|u_0 - \alpha_k| \leq \delta_k, \quad (3.3)$$

With $\varepsilon_k \to 0, \delta_k \to 0$ as $k \to \infty$where f, u_0 and f_k, α_k are as in (1.1) and (3.1). Let n(t) and $\bar{r}$ be as in Lemma 2.2. If $u_k(t)$, (k=1,2,…) and u(t) are respectively solutions of equations (3.1) and (1.1) then as$k \to \infty$, $u_k(t) \to u(t)$.

Proof. For k=1,2,… the conditions of Lemma 2.2 hold. By applying the Lemma 2.3 we have

$$|u_k(t) - u(t)| \leq \frac{\varepsilon_k t + \delta_k}{1-\bar{r}} e_{n+g}(t, t_0)\Delta t \quad (3.4)$$

for k=1,2,…, $t \in I_T$.

CONCLUSION

To summarize, it is conclude that result in Lemma 3.1 gives sufficient solutions works on integro-differential equations

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