INTEGRAL INVOLVING EXTENDED JACOBI POLYNOMIAL, I-FUNCTION AND GENERAL CLASS OF POLYNOMIALS

Dr. Pawan Agrawal
Department of Mathematics, Raj Rishi College, Alwar (Rajasthan)

Abstract—in this paper we establish finite integral which are believed to be new. Our integral involve the product of the extended Jacobi polynomials, I-function and general class of polynomials, on account of the general nature of the function and polynomials occurring in the integral our findings provide interesting extensions of a large number of results.

I. INTRODUCTION: To unify the classical orthogonal polynomials viz. Jacobi, Hermite and Laguerre Fujiwara [2] defined a class of generalized classical polynomials by means of following Rodrigues formula:

\[
R_n(x) = \frac{(-1)^n n!}{n!(x-p)^n (q-x)^n} \frac{d^n}{dx^n}[(x - p)^{\beta + n} (q - x)^{\alpha + n}], \text{ } p < x < q, \text{ } \alpha > -1, \beta > -1
\]  

(1.1)

Denote these polynomials by \( F_n(\beta, \alpha; x) \) and call them extended Jacobi polynomials. Thakare [9] obtained the following form of \( R_n(x) = F_n(\beta, \alpha; x) \)

\[
F_n(\beta, \alpha; x) = (-1)^n \frac{n!}{(n\alpha)_n (n\beta)_n} 2F1 \left[ \begin{array}{c} -n, -n - \alpha; \frac{p-x}{q-x} \\ 1 + \beta \end{array} \right] p < x < q
\]  

(1.2)

Fujiwara [2] proved that when \( p = 1, q = 1 \) and \( k = \frac{1}{2} \)

\[
F_n(\beta, \alpha; x) = \frac{(-1)^n}{n!} (1-x)_2F1 \left[ \begin{array}{c} -n, -n - \alpha; \frac{1}{x-1} \\ 1 + \beta \end{array} \right]
\]

(1.3)

Saxena [5] introduced the I-function defined as:

\[
I_{\alpha, \beta}^{\gamma}(x) = \frac{\prod_{i=1}^{n} (\alpha - a_i) (\alpha - b_i)}{\prod_{i=1}^{n} (\beta - a_i) (\beta - b_i)}
\]  

(1.4)

For \( R = 1 \), the I-Fucntion reduces to the well-known Fox’s H-funtion [6]

A general class of polynomials [7, p. 1, eq. (1)]

\[
S_{N}[x] = \sum_{k=0}^{N} \frac{(-N)_{M}A_{N,k} \alpha^{k}}{k!} x^{k}, \text{ } (N=0,1,2,\ldots)
\]  

(1.5)

where \( M \) is an arbitrary positive integer and the coefficient \( A_{N,k} \) are arbitrary constants real or complex. On suitably specializing the coefficient \( A_{N,k} \), \( S_{N}[x] \) yields a number of known polynomials as its special cases. These include, among others, Laguerre polynomials, Hermite polynomials and several others [8, pp. 158-161].

PRELIMINARIES

In this paper we need the following results:

(i) \[ (i) \] p.10, eq.(13) \( \int_{0}^{1} (t-b)^{a-1}(a-t)^{b-1} dt = (a-b)^{a+b-1} B(x,y), \text{ } Re(x) > 0, \text{ } Re(y) > 0, b < a \)

(ii) The Hyper Geometric function \[ (ii) \] \[ \sum_{n=0}^{\infty} \frac{(a)_{n} (b)_{n} x^{n}}{(c)_{n} n!} \]

(iii) Vandermonde’s theorem [3]

\[ (iv) \] The following results:

\[ (iv) \] \[ \frac{(a)_{n}}{(a+n)} \]

and

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Main Integral:
\[
\int_{p}^{q} (x-p)^{r}(q-x)^{a} F_{\alpha}(\beta, \alpha; x) I_{p,q,R}^{m,n}[z(x-p)^{h}]
\]
\[
I_{p,q,R}^{m,n}[z(x-p)^{h}] = \begin{cases}
(a_{j}, a_{j})_{1,n}; & (a_{j}, a_{j})_{m+1,r} \\
(b_{j}, b_{j})_{1,m}; & (b_{j}, b_{j})_{m+1,q} \end{cases}
\]
\[
S_{N_{1}}^{M_{1}}[e(x-p)^{r}] dx
\]
\[
k^{h} \frac{n!}{(n + \alpha + 1)} \sum_{k_{1}=0}^{[N_{1}/M_{1}]} \frac{(-N_{1})M_{1}A_{N_{1}k_{1}}}{k_{1}!} e^{k_{1}(q-p)^{t+a+n+1-vk_{1}}}
\]
\[
I_{p,t+2,a_{j}+2,R}^{m,n+2}[z(q-p)^{h}]
\]
\[
= \begin{cases}
(-t - vk_{1}, h); & (\beta - t, h)(a_{j}, a_{j})_{1,n}; (a_{j}, a_{j})_{m+1,r} \\
(b_{j}, b_{j})_{1,m}; & (b_{j}, b_{j})_{m+1,q}; (\beta - t + n, h); (-1 - n - t - \alpha - vk_{1}, h) \end{cases}
\]
(3.1)

Provided that Re (\alpha) > -1, Re (t + h\frac{b}{\beta}) > -1, h > 0, j=1, ..., m,

To establish (3.1) replace I-function by its Mellin-Barnes contour integral form (1.4) and get following form of integral (say \Delta)
\[
\Delta = \int_{p}^{q} (x-p)^{t}(q-x)^{a} F_{\alpha}(\beta, \alpha, x) S_{N_{1}}^{M_{1}}[e(x-p)^{r}] dx
\]
(3.2)
\[
= \frac{1}{2\pi i} \int_{L} \phi(\xi) x^{\xi} (x-p)^{h\xi} d\xi
\]

Now interchanging the order of integration which is justified due to absolute convergence of integral involved in the process, we get :
\[
\Delta = \frac{1}{2\pi i} \int_{L} \phi(\xi) x^{\xi} \left[ \int_{p}^{q} (x-p)^{t+h\xi}(q-x)^{a} F_{\alpha}(\beta, \alpha, x) S_{N_{1}}^{M_{1}}[e(x-p)^{r}] dx \right] d\xi
\]
(3.3)

Now we put value of \(F_{\alpha}(\beta, \alpha, x)\) from (1.2) and value of \(S_{N_{1}}^{M_{1}}[e(x-p)^{r}]\) from (1.5) in (3.3) and interchange the order of integration and summation and using the result (2.1) to (2.5) and after little simplification it becomes:
\[
\Delta = \sum_{k_{1}=0}^{[N_{1}/M_{1}]} \frac{(-N_{1})M_{1}A_{N_{1}k_{1}}}{k_{1}!} e^{k_{1}(q-p)^{t+a+n+1-vk_{1}}} \frac{n!}{k_{1}!} \Gamma(n + \alpha + 1)
\]
(3.4)

Now using definition of I-function, we obtain reqd. result (3.1).

Special Cases:
In the main result if we take \(N_{1}=0\) (the polynomial \(S_{0}^{M_{1}}\) will reduce to \(A_{0,0}\) Which can be taken to be unity without loss of generality), we arrive at a result given by S. C. Sharma [4. eq. (3.1)].

REFERENCES