Interval Valued Intuitionistic (S, T)-Fuzzy Hv-Submodules and their Characterizations

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Abstract: The notion of interval valued intuitionistic fuzzy Hv-submodules of an Hv-module with respect to a t-norm T and an s-norm S is given by J.M. Zhan. In this paper, we give some results on interval valued intuitionistic (S, T)-fuzzy Hv-submodules of an Hv-modules.

Keywords: Hv-module, interval valued intuitionistic (S, T)-fuzzy Hv-submodule, interval valued intuitionistic (S, T)-fuzzy relation.


1. Introduction

The concept of hyperstructure was introduced in 1934 by Marty [1]. Hyperstructures have many applications to several branches of pure and applied sciences. Vougiouklis [2] introduced the notion of H_v-structures, and Davvaz [3] surveyed the theory of H_v-structures. After the introduction of fuzzy sets by Zadeh [4], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [5] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [6, 7].


Basing on [11], in this paper, we apply the notion of interval valued intuitionistic (S, T)-fuzzy Hv-submodules of an Hv-module and describe the characteristic properties. The paper is organized as follows: in section 2 some fundamental definitions on Hv-structures and fuzzy sets are explored, in section 3 we establish some useful properties on interval valued intuitionistic (S, T)-fuzzy Hv-submodules and in section 4 interval valued intuitionistic (S, T)-fuzzy relations on an Hv-module are discussed.

2. Basic Definitions

We first give some basic definitions for proving the further results.

Definition 2.1 [12] Let X be a non-empty set. A mapping \( \mu : X \rightarrow [0, 1] \) is called a fuzzy set in X. The complement of \( \mu \), denoted by \( \mu^c \), is the fuzzy set in X given by
\[
\mu^c(x) = 1 - \mu(x) \quad \forall x \in X.
\]

Definition 2.2 [12] Let \( f \) be a mapping from a set X to a set Y. Let \( \mu \) be a fuzzy set in X, and \( \lambda \) be a fuzzy set in Y. Then the inverse image \( f^{-1}(\lambda) \) of \( \lambda \) is a fuzzy set in X defined by
\[
f^{-1}(\lambda)(x) = \lambda(f(x)) \quad \forall x \in X.
\]
The image \( f(\mu) \) of \( \mu \) is the fuzzy set in Y defined by
\[
f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}
\]
For all \( y \in Y \).

Definition 2.3 [12] An intuitionistic fuzzy set A in a non-empty set X is an object having the form
\[ A = \{(x, \alpha_A(x), \beta_A(x)) : x \in X\}, \]
where the functions \( \alpha_A : X \rightarrow [0, 1] \) and \( \beta_A : X \rightarrow [0, 1] \) denote the degree of membership and degree of non membership of each element \( x \in X \) to the set A respectively and \( 0 \leq \alpha_A(x) + \beta_A(x) \leq 1 \) for all \( x \in X \). We shall use the symbol \( A = \{\alpha_A, \beta_A\} \) for the intuitionistic fuzzy set \( A = \{(x, \alpha_A(x), \beta_A(x)) : x \in X\} \).

Definition 2.4 [12] Let \( A = \{\alpha_A, \beta_A\} \) and \( B = \{\alpha_B, \beta_B\} \) be intuitionistic fuzzy sets in X. Then
\[ (1) \quad A \subseteq B \iff \alpha_A(x) \leq \alpha_B(x) \quad \text{and} \quad \beta_A(x) \leq \beta_B(x), \]
\[ (2) \quad A = B \iff \alpha_A(x) = \alpha_B(x) \quad \text{and} \quad \beta_A(x) = \beta_B(x). \]
(2) $A' = \{ (x, \beta (x), \alpha (x)) : x \in X \}$,
(3) $A \cap B = \{ (x, \min (\alpha (x), \alpha (b(x))), \\
\max (\beta (x), \beta (b(x))) : x \in X \}$,
(4) $A \cup B = \{ (x, \max (\alpha (x), \alpha (b(x))), \\
\min (\beta (x), \beta (b(x))) : x \in X \}$,
(5) $\square A = \{ (x, \alpha (x), \alpha (x)) : x \in X \}$,
(6) $\triangle A = \{ (x, \beta (x), \beta (x)) : x \in X \}$.

**Definition 2.5** [13] Let $G$ be a non-empty set and $*: G \times G \rightarrow \wp^*(G)$ be a hyperoperation, where $\wp^*(G)$ is the set of all the non-empty subsets of $G$. Where $A*B = \bigcup_{a \in A, b \in B} a*b$, $\forall A, B \subseteq G$.

The $*$ is called weak commutative if $x*y \cap y*x \neq \phi$, $\forall x, y \in G$.

The $*$ is called weak associative if $(x*y)*z \cap x*(y*z) \neq \phi$, $\forall x, y, z \in G$.

A hyperstructure $(G, *)$ is called an $H_r$-group if

(i) $*$ is weak associative.
(ii) $a * G = G * a = G$, $\forall a \in G$ (Reproduction axiom).

**Definition 2.6** [14] Let $G$ be a $H_r$-group and let $\mu$ be a fuzzy subset of $G$. Then $\mu$ is said to be a fuzzy $H_r$-subgroup of $G$ if the following axioms hold:

(i) $\min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z)\}$, $\forall x, y \in G$ (ii) For all $x, a \in G$ there exists $y \in G$ such that $x \in a*y$ and $\min\{\mu(a), \mu(x)\} \leq \{\mu(y)\}$.

**Definition 2.7** [15] Let $G$ be a $H_r$-group. An intuitionistic fuzzy set $A = [\alpha_A, \beta_A]$ of $G$ is called intuitionistic fuzzy $H_r$-subgroup of $G$ if the following axioms hold:

(i) $\min\{\alpha_A(x), \alpha_A(y)\} \leq \inf\{\alpha_A(z)\}$, $\forall x, y \in G$.
(ii) For all $x, a \in G$ there exists $y \in G$ such that $x \in a*y$ and $\min\{\alpha_A(a), \alpha_A(x)\} \leq \{\alpha_A(y)\}$.
(iii) $\sup\{\beta_A(z)\} \leq \max\{\beta_A(x), \beta_A(y)\}$, $\forall x, y \in G$.
(iv) For all $x, a \in G$ there exists $y \in G$ such that $x \in a*y$ and $\{\beta_A(y)\} \leq \max\{\beta_A(a), \beta_A(x)\}$.

**Definition 2.8** [13] An $H_r$-ring is a system $(R, +, \cdot)$ with two hyperoperations satisfying the ring-like axioms:

(i) $(R, +)$ is an $H_r$-group, that is,

$((x+y)+z) \cap (x+(y+z)) \neq \phi$, $\forall x, y, z \in R$,

$a + R = R + a = R$, $\forall a \in R$;

(ii) $(R, \cdot)$ is an $H_r$-semigroup;

(iii) $(\cdot)$ is weak distributive with respect to $(+)$, that is, for all $x, y, z \in R$,

$(x \cdot (y+z)) \cap (x \cdot y + x \cdot z) \neq \phi$,

$((x+y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \phi$.

**Definition 2.9** [16] Let $R$ be an $H_r$-ring. A nonempty subset $I$ of $R$ is called a left (resp., right) $H_r$-ideal if the following axioms hold:

(i) $(I, +)$ is an $H_r$-subgroup of $(R, +)$.

(ii) $R \cdot I \subseteq I$ (resp., $I \cdot R \subseteq I$).
Definition 2.10 [16] Let \((R, +, \cdot)\) be an \(H_r\)-ring and \(\mu\) a fuzzy subset of \(R\). Then \(\mu\) is said to be a left (resp., right) fuzzy \(H_r\)-ideal of \(R\) if the following axioms hold:

1. \(\min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) : z \in x + y\} \forall x, y \in R\),

2. For all \(x, a \in R\) there exists \(y \in R\) such that \(x \in a + y\) and \(\min\{\mu(a), \mu(x)\} \leq \mu(y)\),

3. For all \(x, a \in R\) there exists \(z \in R\) such that \(x \in z + a\) and \(\min\{\mu(a), \mu(x)\} \leq \mu(z)\),

4. \(\mu(y) \leq \inf\{\mu(z) : z \in x \cdot y\}\) [respectively \(\mu(x) \leq \inf\{\mu(z) : z \in x \cdot y\}\)] \(\forall x, y \in R\) .

Definition 2.11 [16] An intuitionistic fuzzy set \(A = \{\alpha_A, \beta_A\}\) in \(R\) is called a left (resp., right) intuitionistic fuzzy \(H_r\)-ideal of \(R\) if the following axioms hold:

1. \(\min\{\alpha_A(x), \alpha_A(y)\} \leq \inf\{\alpha_A(z) : z \in x + y\}\) and \(\max\{\beta_A(x), \beta_A(y)\} \geq \sup\{\beta_A(z) : z \in x + y\}\) for all \(x, y \in R\),

2. For all \(x, a \in R\) there exists \(y \in R\) such that \(x \in a + y\) and \(\min\{\alpha_A(a), \alpha_A(x)\} \leq \alpha_A(z)\) and \(\max\{\beta_A(a), \beta_A(x)\} \geq \beta_A(y)\),

3. For all \(x, a \in R\) there exists \(z \in R\) such that \(x \in z + a\) and \(\min\{\mu(a), \mu(x)\} \leq \mu(z)\) and \(\max\{\beta_A(a), \beta_A(x)\} \geq \beta_A(z)\),

4. \(\alpha_A(y) \leq \inf\{\alpha_A(z) : z \in x \cdot y\}\) [respectively \(\alpha_A(x) \leq \inf\{\alpha_A(z) : z \in x \cdot y\}\)] \(\forall x, y \in R\) and \(\beta_A(y) \geq \sup\{\beta_A(z) : z \in x \cdot y\}\) [respectively \(\beta_A(x) \geq \sup\{\beta_A(z) : z \in x \cdot y\}\)] \(\forall x, y \in R\) .

Definition 2.12 [16] A nonempty set \(M\) is called an \(H_r\)-module over an \(H_r\)-ring \(R\) if \((M, +)\) is a weak commutative \(H_r\)-group and there exists a map \(\phi : R \times M \rightarrow \phi^*(M), (r, x) \rightarrow r.x\) such that for all \(a, b \in R\) and \(x, y \in M\), we have

\[
(a \cdot (x + y)) \cap (a \cdot x + a \cdot y) \neq \phi,
\]

\[
((x + y) \cdot a) \cap (x \cdot a + y \cdot a) \neq \phi,
\]

\[
(a \cdot (b \cdot x)) \cap ((a \cdot b) \cdot x) \neq \phi.
\]

Note that by using fuzzy sets, we can consider the structure of \(H_r\)-module on any ordinary module which is a generalization of a module.

Definition 2.13 [18] A fuzzy set \(\mu\) in \(M\) is called a fuzzy \(H_r\)-submodule of \(M\) if

1. \(\min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) : z \in x + y\}\) \(\forall x, y \in M\),

2. For all \(x, a \in M\) there exists \(y, z \in M\) such that \(x \in (a + y) \cap (z + a)\) and \(\min\{\mu(a), \mu(x)\} \leq \inf\{\mu(y), \mu(z)\}\),

3. \(\mu(y) \leq \inf\{\mu(z) : z \in x \cdot y\}\) for all \(y \in M\) and \(x \in R\).

Definition 2.14 [11] An intuitionistic fuzzy set \(A = \{\alpha_A, \beta_A\}\) in an \(H_r\)-module \(M\) over an \(H_r\)-ring \(R\) is said to be an intuitionistic fuzzy \(H_r\)-submodule of \(M\) if the following axioms hold:

1. \(\min\{\alpha_A(x), \alpha_A(y)\} \leq \inf\{\alpha_A(z) : z \in x + y\}\) and \(\max\{\beta_A(x), \beta_A(y)\} \geq \sup\{\beta_A(z) : z \in x + y\}\) for all \(x, y \in M\),

2. For all \(x, a \in M\) there exists \(y \in M\) such that \(x \in a + y\) and \(\min\{\alpha_A(a), \alpha_A(x)\} \leq \alpha_A(z)\) and \(\max\{\beta_A(a), \beta_A(x)\} \geq \beta_A(y)\),
(3) For all \(x,a \in M\) there exists \(z \in M\) such that \(x \in z + a\) and \(\min\{\mu_A(a), \mu_A(x)\} \leq \mu_A(z)\) and \(\max\{\beta_A(a), \beta_A(x)\} \geq \beta_A(z)\),

(4) \(\alpha_A(x) \leq \inf\{\alpha_A(z) : z \in r \cdot x\}\) and \(\beta_A(x) \geq \sup\{\beta_A(z) : z \in r \cdot x\}\) for all \(x \in M\) and \(r \in R\).

**Definition 2.15** [17] By a \(t\)-norm \(T\), we mean a function \(T : [0, 1] \times [0, 1] \to [0, 1]\) satisfying the following conditions:

(i) \(T(x, 1) = x\),

(ii) \(T(x, y) \leq T(x, z)\) if \(y \leq z\),

(iii) \(T(x, y) = T(y, x)\),

(iv) \(T(x, T(y, z)) = T(T(x, y), z)\)

For all \(x, y, z \in [0, 1]\).

**Definition 2.16** [17] By a \(s\)-norm \(S\), we mean a function \(S : [0, 1] \times [0, 1] \to [0, 1]\) satisfying the following conditions:

(i) \(S(x, 0) = x\),

(ii) \(S(x, y) \leq S(x, z)\) if \(y \leq z\),

(iii) \(S(x, y) = S(y, x)\),

(iv) \(S(x, S(y, z)) = S(S(x, y), z)\)

For all \(x, y, z \in [0, 1]\).

It is clear that \(T(\alpha, \beta) \leq \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq S(\alpha, \beta)\) for all \(\alpha, \beta \in [0, 1]\).

By an interval number \(\tilde{a}\), we mean an interval \([a^-, a^+]\) where \(0 \leq a^- \leq a^+ \leq 1\). The set of all interval numbers is denoted by \(D[0, 1]\). We also identify the interval \([a, a]\) by the number \(a \in [0, 1]\).

For the interval numbers \(\tilde{a}_i = [a^-_i, a^+_i] \in D[0, 1], i \in I\), we define

\[
\begin{align*}
\max\{\tilde{a}_i, \tilde{b}_i\} &= \left[\max\{a^-_i, b^-_i\}, \max\{a^+_i, b^+_i\}\right], \\
\min\{\tilde{a}_i, \tilde{b}_i\} &= \left[\min\{a^-_i, b^-_i\}, \min\{a^+_i, b^+_i\}\right], \\
\inf \tilde{a}_i &= \left[\bigwedge_{i \in I} a^-_i, \bigwedge_{i \in I} a^+_i\right], \\
\sup \tilde{a}_i &= \left[\bigvee_{i \in I} a^-_i, \bigvee_{i \in I} a^+_i\right]
\end{align*}
\]

and put

(1) \(\tilde{a}_1 \leq \tilde{a}_2 \iff a^-_1 \leq a^-_2\) and \(a^+_1 \leq a^+_2\),

(2) \(\tilde{a}_1 = \tilde{a}_2 \iff a^-_1 = a^-_2\) and \(a^+_1 = a^+_2\),

(3) \(\tilde{a}_1 < \tilde{a}_2 \iff \tilde{a}_1 \leq \tilde{a}_2\) and \(\tilde{a}_1 \neq \tilde{a}_2\),

(4) \(k\tilde{a} = [ka^-, ka^+]\), whenever \(0 \leq k \leq 1\).

It is clear that \((D[0, 1], \leq, \vee, \wedge)\) is a complete lattice with \(0 = [0, 0]\) as least element and \(1 = [1, 1]\) as greatest element.

By an interval valued fuzzy set \(F\) on \(X\) we mean the set \(F = \{(x, [\alpha^-_F(x), \alpha^+_F(x)] : x \in X\}\). Where \(\alpha^-_F\) and \(\alpha^+_F\) are fuzzy subsets of \(X\) such that \(\alpha^-_F(x) \leq \alpha^+_F(x)\) for all \(x \in X\). Put \(\tilde{\alpha}_F(x) = [\alpha^-_F(x), \alpha^+_F(x)]\). Then \(F = \{(x, \tilde{\alpha}_F(x)) : x \in X\}\), where \(\tilde{\alpha}_F : X \to D[0, 1]\).

If \(A, B\) are two interval valued fuzzy subsets of \(X\), then we define \(A \subseteq B\) if and only if for all \(x \in X\), \(\alpha^-_A(x) \leq \alpha^-_B(x)\) and \(\alpha^+_A(x) \leq \alpha^+_B(x)\).
A = B if and only if for all $x \in X$, $\alpha_A^{-}(x) = \alpha_B^{-}(x)$ and $\alpha_A^{+}(x) = \alpha_B^{+}(x)$.

Also, the union, intersection and complement are defined as follows: let $A; B$ be two interval valued fuzzy subsets of $X$, then

$A \cup B = \left\{(x, \max \{\alpha_A(x), \alpha_B(x)\}, \max \{\alpha_A^{+}(x), \alpha_B^{+}(x)\}) : x \in X\right\},$

$A \cap B = \left\{(x, \min \{\alpha_A(x), \alpha_B(x)\}, \min \{\alpha_A^{+}(x), \alpha_B^{+}(x)\}) : x \in X\right\},$

$A^{c} = \left\{(x, [1-\alpha_A(x), 1-\alpha_A^{+}(x)]) : x \in X\right\}.$

According to Atanassov an interval valued intuitionistic fuzzy set on $X$ is defined as an object of the form $A = \{(\alpha_A, \beta_A) : x \in X\}$, where $\alpha_A(x)$ and $\beta_A(x)$ are interval valued fuzzy sets on $X$ such that $0 \leq \sup \alpha_A(x) + \sup \beta_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, in the following such interval valued intuitionistic fuzzy sets will be denoted by $A = (\alpha_A, \beta_A)$.

3. Interval Valued Intuitionistic (S, T)-Fuzzy Hv-Submodules
In what follows, let $M$ denote an Hv-module over an Hv-ring $R$ unless otherwise.

**Definition 3.1.** An interval valued intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ of $M$ is called an intuitionistic fuzzy Hv-submodule of $M$ with respect to t-norm $T$ and s-norm $S$ (briefly, intuitionistic (S, T)-fuzzy Hv-submodule of $M$) if it satisfies the following conditions:

(1) $T(\alpha_A(x), \alpha_A(y)) \leq \inf_{x \leq y} \alpha_A(z)$ and $S(\alpha_A(x), \beta_A(y)) \geq \sup_{x \leq y} \beta_A(z)$, $\forall x, y \in M$.

(2) For all $x, a \in M$ there exists $y \in M$ such that $x = a + y$ and $T(\alpha_A(a), \alpha_A(x)) \leq \alpha_A(y)$ and $S(\beta_A(a), \beta_A(x)) \geq \beta_A(y)$.

(3) For all $x, a \in M$ there exists $z \in M$ such that $x = z + a$ and $T(\alpha_A(a), \alpha_A(x)) \leq \alpha_A(z)$ and $S(\beta_A(a), \beta_A(x)) \geq \beta_A(z)$.

(4) $\alpha_A(x) \leq \inf_{x \leq r} \alpha_A(z)$ and $\beta_A(x) \geq \sup_{x \leq r} \beta_A(z)$, for all $x \in M$ and $r \in R$.

**Definition 3.2.** The norms $T$ and $S$ are called dual if for all $a,b \in [0,1], T(a,b) = S(a,b)$.

**Lemma 3.3.** Let $T$ and $S$ be dual norms. If $A = (\alpha_A, \beta_A)$ is an interval valued intuitionistic (S, T)-fuzzy Hv-submodule of $M$, then so is $\overline{\Delta}A = (\overline{\alpha_A}, \overline{\beta_A})$.

**Proof.** It is sufficient to show that $\overline{\alpha_A}$ satisfies the conditions of Definition 3.1. For all $x, y \in M$, we have $T(\overline{\alpha_A}(x), \overline{\alpha_A}(y)) \leq \inf_{x \leq y} \overline{\alpha_A}(z)$ and so $T(1-\overline{\alpha_A}(x), 1-\overline{\alpha_A}(y)) \leq \inf_{x \leq y} (1-\overline{\alpha_A}(z))$.

Hence $T(1-\overline{\alpha_A}(x), 1-\overline{\alpha_A}(y)) \leq \inf_{x \leq y} (1-\overline{\alpha_A}(z))$.

Which implies $T(1-\overline{\alpha_A}(x), 1-\overline{\alpha_A}(y)) \leq 1 - \sup_{x \leq y} \overline{\alpha_A}(z)$ since $T$ and $S$ are dual.

Now, let $a, x \in M$. Then there exists $y \in M$ such that $x = a + y$ and $T(\overline{\alpha_A}(a), \overline{\alpha_A}(x)) \leq \overline{\alpha_A}(y)$. It follows that that $T(1-\overline{\alpha_A}(a), 1-\overline{\alpha_A}(x)) \leq 1 - \overline{\alpha_A}(y)$, so that $\overline{\alpha_A}(y) \leq 1 - T(1-\overline{\alpha_A}(a), 1-\overline{\alpha_A}(x)) = S(\overline{\alpha_A}(a), \overline{\alpha_A}(x))$.

Similarly, let $a, x \in M$. Then there exists $z \in M$ such that $x = z + a$ and $\overline{\alpha_A}(z) \leq S(\overline{\alpha_A}(a), \overline{\alpha_A}(x))$.

Now, let $x \in M$ and $r \in R$, we have $\overline{\alpha_A}(x) \leq \inf_{x \leq r} \overline{\alpha_A}(z)$ since $\overline{\alpha_A}$ is a $T$-fuzzy Hv-submodule of $M$. Hence $1-\overline{\alpha_A}(x) \leq \inf_{x \leq r} (1-\overline{\alpha_A}(z))$ which implies $\sup_{x \leq r} \overline{\alpha_A}(z) \leq \overline{\alpha_A}(x)$. Therefore $\overline{\Delta}A = (\overline{\alpha_A}, \overline{\beta_A})$ is an intuitionistic (S, T)-fuzzy Hv-submodule of $M$. 
Lemma 3.4. Let $T$ and $S$ be dual norms. If $A = (\tilde{a}, \tilde{a})$ is an interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule of $M$, then so is $\Diamond A = (\overline{\tilde{b}}, \overline{\tilde{b}})$.

**Proof.** The proof is similar to the proof of Lemma 3.3. Combining the above two lemmas it is not difficult to verify that the following theorem is valid.

**Theorem 3.5.** Let $T$ and $S$ be dual norms. Then $A = (\tilde{a}, \tilde{a})$ is an interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule of $M$ if and only if $\Box A$ and $\Diamond A$ are interval valued intuitionistic $(S, T)$-fuzzy Hv-submodules of $M$.

**Corollary 3.6.** Let $T$ and $S$ be dual norms. Then $A = (\tilde{a}, \tilde{a})$ is an interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule of $M$ if and only if $\tilde{a}$ and $\tilde{a}$ satisfy the imaginable property.

**Definition 3.7.** An interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule $A = (\tilde{a}, \tilde{a})$ of $M$ is said to be imaginable if $\tilde{a}$ and $\tilde{a}$ satisfy the imaginable property.

The following are obvious.

**Lemma 3.8.** Every imaginable interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule of $M$ is interval valued intuitionistic fuzzy Hv-submodule.

**Lemma 3.9.** [19] A fuzzy set $\mu$ in $M$ is a fuzzy Hv-submodule of $M$ if and only if the non-empty $U(\mu; \alpha)$, $\alpha \in [0, 1]$ is an Hv-submodule of $M$.

**Lemma 3.10.** [19] A fuzzy set $\mu$ in $M$ is a fuzzy Hv-submodule of $M$ if and only if the non-empty $\mu$ is an anti-fuzzy Hv-submodule of $M$.

By the above Lemmas, we can give the following results.

**Theorem 3.11.** If $A = (\tilde{a}, \tilde{a})$ is an imaginable interval valued intuitionistic fuzzy set in $M$. Then $A = (\tilde{a}, \tilde{a})$ is an interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule of $M$ if and only if the non-empty sets $U(\tilde{a}; \alpha)$ and $L(\tilde{a}; \alpha)$ are Hv-submodules of $M$, for every $\alpha \in [0, 1]$.

**Theorem 3.12.** Let $A = (\tilde{a}, \tilde{a})$ be an interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule of $M$. Then $\tilde{a}(x) = \sup\{\alpha \in [0, 1] | x \in U(\tilde{a}; \alpha)\}$ and $\tilde{a}(x) = \inf\{\alpha \in [0, 1] | x \in L(\tilde{a}; \alpha)\}$, for all $x \in M$.

**Definition 3.13.** Let $f: M \rightarrow M'$ be a strong epimorphism of Hv-modules. If $A = (\tilde{a}, \tilde{a})$ is an interval valued intuitionistic fuzzy set in $M'$, then the inverse image of $A$ under $f$, denoted by $f^{-1}(A)$, is an interval valued intuitionistic fuzzy set in $M$, defined by $f^{-1}(A) = (f^{-1}(\tilde{a}), f^{-1}(\tilde{a}))$.

By the above Definition, we can give the following result.

**Theorem 3.14.** Let $f: M \rightarrow M'$ be a strong epimorphism of Hv-modules. If $A = (\tilde{a}, \tilde{a})$ is an interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule of $M$. Then the inverse image $f^{-1}(A) = (f^{-1}(\tilde{a}), f^{-1}(\tilde{a}))$ of $A$ under $f$ is an interval valued intuitionistic $(S, T)$-fuzzy Hv-submodule of $M$.

4. Interval Valued Intuitionistic $(S, T)$-Fuzzy Relations

We first recall that a fuzzy relation on any set $X$ is a fuzzy set $\mu: X \times X \rightarrow [0, 1]$. We now give the following definitions and cite some known results.

**Definition 4.1.** An interval valued intuitionistic fuzzy set $A = (\tilde{a}, \tilde{a})$ is called an interval valued intuitionistic fuzzy relation on any set $X$ if $\tilde{a}$ and $\tilde{a}$ are fuzzy relations on $X$.

**Definition 4.2.** Let $A = (\tilde{a}, \tilde{b})$ and $B = (\tilde{b}, \tilde{b})$ be interval valued intuitionistic fuzzy sets on a set $X$. If $A = (\tilde{a}, \tilde{a})$ is an interval valued intuitionistic fuzzy relation on $X$, then $A = (\tilde{a}, \tilde{b})$ is called an interval valued intuitionistic $(S, T)$-fuzzy relation on $B = (\tilde{b}, \tilde{b})$ if and $\tilde{b}(x, y) \geq S(\tilde{b}(x), \tilde{b}(y))$, for all $x, y \in X$.

**Definition 4.3.** The interval valued intuitionistic $(S, T)$-Cartesian product of $A$ and $B$, denoted by $A \times B$, is an interval valued intuitionistic fuzzy set on $X$, which is defined by $A \times B = (\tilde{a}, \tilde{b}) \times (\tilde{b}, \tilde{b}) = (\tilde{a} \times \tilde{b}, \tilde{b} \times \tilde{b})$, where $(\tilde{a} \times \tilde{b})(x, y) = T(\tilde{a}(x), \tilde{b}(y))$ and $(\tilde{b} \times \tilde{b})(x, y) = S(\tilde{b}(x), \tilde{b}(y))$ hold for all $x, y \in X$.

**Lemma 4.4.** If $A = (\tilde{a}, \tilde{a})$ and $B = (\tilde{b}, \tilde{b})$ are interval valued intuitionistic fuzzy sets on a set $X$. Then we have

(i) $A \times B$ is an interval valued intuitionistic $(S, T)$-fuzzy relation on $X$;
(ii) \( U(\bar{a}_A \times \bar{a}_B; \alpha) = U(\bar{a}_A; \alpha) \times U(\bar{a}_B; \alpha) \) and \( U(\bar{b}_A \times \bar{b}_B; \alpha) = U(\bar{b}_A; \alpha) \times U(\bar{b}_B; \alpha) \) for all \( \alpha \in [0,1] \).

**Definition 4.5.** If \( A = \left( \bar{a}_A, \bar{b}_A \right) \) and \( B = \left( \bar{a}_B, \bar{b}_B \right) \) are interval valued intuitionistic fuzzy sets on a set \( X \), the strongest interval valued intuitionistic \((S, T)\)-fuzzy relation on \( X \) is defined by \( A_B = (\bar{a}_{A_{\text{sup}}}, \bar{b}_{A_{\text{sup}}}) \), where \( \bar{a}_{A_{\text{sup}}}(x, y) = T(\bar{a}_B(x), \bar{a}_B(y)) \) and \( \bar{b}_{A_{\text{sup}}}(x, y) = S(\bar{b}_B(x), \bar{b}_B(y)) \) for all \( x, y \in X \).

**Lemma 4.6.** For the interval valued intuitionistic fuzzy sets \( A = \left( \bar{a}_A, \bar{b}_A \right) \) and \( B = \left( \bar{a}_B, \bar{b}_B \right) \) on a set \( X \), let \( A_B \) be the strongest interval valued intuitionistic \((S, T)\)-fuzzy relation on \( X \). Then for any \( \alpha \in [0,1] \), we have \( U(\bar{a}_{A_{\text{sup}}}; \alpha) = U(\bar{a}_B; \alpha) \times U(\bar{a}_B; \alpha) \) and \( L(\bar{b}_{A_{\text{sup}}}; \alpha) = L(\bar{b}_B; \alpha) \times L(\bar{b}_B; \alpha) \).

**Lemma 4.7.** [20] For all \( \alpha, \beta, \delta, \gamma \in [0, 1] \), we have \( T(\bar{a}(\alpha), \bar{b}(\beta), \bar{c}(\gamma), \bar{d}(\delta)) = T(\bar{a}(\alpha), \bar{b}(\beta), \bar{c}(\gamma), \bar{d}(\delta)) \). Then we can verify the following conditions of definition 3.1.

1. Let \( x = (x_1, x_2), y = (y_1, y_2) \in M \times M \). For any \( z = (z_1, z_2) \in x + y \), we have
   \[
   \inf_{z \in x+y} \bar{a}_{A_{\text{inf}}}(z) = \inf_{(x_1, x_2) \in (x_1+y_1, y_1+y_2)} (\inf_{z \in x+y} \bar{a}_{A_{\text{inf}}}(z_1, z_2)) = T(\inf_{(x_1, x_2) \in (x_1+y_1, y_1+y_2)} \bar{a}_{A_{\text{inf}}}(z_1, z_2)) \geq T(\bar{a}_B(x_1), \bar{a}_B(y_1), \bar{a}_B(x_2), \bar{a}_B(y_2)) \]
   \[
   = T(\bar{a}_B(x_1), \bar{a}_B(x_2), \bar{a}_B(y_1), \bar{a}_B(y_2)) = T(\bar{a}_B(x_1, x_2), \bar{a}_B(y_1, y_2)) = T(\bar{a}_B(x_1), \bar{a}_B(y_1), \bar{a}_B(x_2), \bar{a}_B(y_2)) = T(\bar{a}_A, \bar{a}_B(x_1, x_2), \bar{a}_B(y_1, y_2)) = T(\bar{a}_A, \bar{a}_B(x), \bar{a}_B(y)).
   \]
   Similarly, we have \( \sup_{z \in x+y} \bar{b}_{A_{\text{sup}}}(z) \leq S(\bar{b}_{A_{\text{sup}}}(a), \bar{b}_{A_{\text{sup}}}(b)) \).

2. For all \( x = (x_1, x_2), a = (a_1, a_2) \in M \times M \). Then \( y_1, y_2 \in M \) such that \( x_1 \in a_1 + y_1 \) and \( x_2 \in a_2 + y_2 \), and thus \( x_1, x_2 \in (a_1, y_1) + (a_2, y_2) = (a_1 + a_2) + (y_1, y_2) \). Moreover, we have
   \[
   \bar{a}_{A_{\text{inf}}}(y) = \bar{a}_{A_{\text{inf}}}(y_1, y_2) = T(\bar{a}_B(y_1), \bar{a}_B(y_2)) \geq T(\bar{a}_B(a_1), \bar{a}_B(y_1), \bar{a}_B(a_2), \bar{a}_B(y_2)) = T(\bar{a}_B(a_1), \bar{a}_B(a_2), \bar{a}_B(x_1), \bar{a}_B(y_2)) \]
   \[
   = T(\bar{a}_B, \bar{a}_B(x_1), \bar{a}_B(y_2)) = T(\bar{a}_A, \bar{a}_B(x_1), \bar{a}_B(y_2)).
   \]
   Similarly, \( \bar{b}_{A_{\text{sup}}}(y) \leq S(\bar{b}_{A_{\text{sup}}}(a), \bar{b}_{A_{\text{sup}}}(b)) \).

3. is similar to (2).
Let \( x = (x_1, x_2) \in M \times M \) and \( r = (r_1, r_2) \in R \times R \). For any \( z = (z_1, z_2) \in (r_1, r_2)\langle x_1, x_2 \rangle \), we have
\[
\inf_{z \in r} \alpha_{a_{gb}}(z) = \inf_{(z_1, z_2) \in r} \alpha_{a_{gb}}(z_1, z_2)
\]
\[
= \inf_{(z_1, z_2) \in r} T(\alpha_B(z_1), \alpha_B(z_2))
\]
\[
\geq T(\inf_{x \in z_1} \alpha_B(z_1), \inf_{x \in z_2} \alpha_B(z_2))
\]
\[
\geq T(\alpha_B(x_1), \alpha_B(x_2)) = \alpha_{a_{gb}}(x_1, x_2)
\]
Similarly, \( \sup_{z \in r} \beta_B(z) \leq \beta_B(x) \).

This shows that \( A_B \) is an interval valued intuitionistic \((S, T)\)-fuzzy Hv-submodule of \( M \times M \).

Now, for any \( x = (x_1, x_2) \in M \times M \), we can easily show that \( T(\alpha_{a_{gb}}(x), \alpha_{a_{gb}}(x)) = \alpha_{a_{gb}}(x) \) and
\[
S(\beta_B(x), \beta_B(x)) = \beta_B(x).
\]
Hence, \( A_B \) is an interval valued imaginable intuitionistic \((S, T)\)-fuzzy Hv-submodule of \( M \times M \).

To prove the converse of the theorem, we need prove the conditions (1)-(4) of definition 3.1 hold.

(1) Let \( x, y \in M \). Then we have
\[
\inf_{z \in x+y} \alpha_B(z) = \inf_{z \in x+y} T(\alpha_B(z), \alpha_B(z))
\]
\[
= \inf_{z \in x+y} \alpha_{a_{gb}}(z, z)
\]
\[
= \inf_{(z, z) \in (x, y)} \alpha_{a_{gb}}(z, z)
\]
\[
\geq T(\alpha_{a_{gb}}(x, x), \alpha_{a_{gb}}(y, y))
\]
\[
\geq T(T(\alpha_B(x), \alpha_B(x)), T(\alpha_B(y), \alpha_B(y)))
\]
\[
= T(\alpha_B(x), \alpha_B(y)).
\]
Similarly, we have \( \sup_{z \in x+y} \beta_B(z) \leq S(\beta_B(x), \beta_B(y)) \).

(2) For all \( x, a \in M \), and thus \( (x, x), (a, a) \in M \times M \). Then there exists \( (y, y) \in M \) such that
\( (x, x) \in (a, a) + (y, y) = (a+y, a+y) \). That is, \( x \in a+y \).

Moreover, we have \( \alpha_B(y) = T(\alpha_B(y), \alpha_B(y)) = \alpha_{a_{gb}}(y, y) \)
\[
\geq T(\alpha_{a_{gb}}(a, a), \alpha_{a_{gb}}(x, x))
\]
\[
= T(T(\alpha_B(a), \alpha_B(a)), T(\alpha_B(x), \alpha_B(x)))
\]
\[
= T(\alpha_B(a), \alpha_B(x)).
\]
Similarly, \( \beta_B(y) \leq S(\beta_B(a), \beta_B(x)) \).

(3) is similar to (2).

(4) Let \( x \in M \) and \( r \in R \), we have
\[
\inf_{z \in r} \alpha_B(z) = \inf_{z \in r} T(\alpha_B(z), \alpha_B(z))
\]
\[
= \inf_{(z, z) \in r} \alpha_{a_{gb}}(z, z)
\]
\[
\geq \alpha_{a_{gb}}(x, x)
\]
\[
= T(\alpha_B(x), \alpha_B(x)) = \alpha_B(x).
\]
Similarly, \( \sup_{z \in r} \beta_B(z) \leq \beta_B(x) \).
This shows that conditions (1)-(4) hold and hence \( B = (\tilde{\alpha}_B, \tilde{\beta}_B) \) is an imaginable interval valued intuitionistic (S, T)-fuzzy Hv-submodule of M.

**Definition 4.11.** If \( A = (\tilde{\alpha}_A, \tilde{\beta}_A) \) and \( B = (\tilde{\alpha}_B, \tilde{\beta}_B) \) are imaginable intuitionistic fuzzy sets on any set X, then the intuitionistic (S, T)-product of A and B, denoted by \([A \cdot B]_{(S,T)}\), is defined by

\[
[A \cdot B]_{(S,T)} = \left[ (\tilde{\alpha}_A \cdot \tilde{\alpha}_B), (\tilde{\beta}_A \cdot \tilde{\beta}_B) \right]_{(S,T)},
\]

\[
= \left[ (\tilde{\alpha}_A \cdot \tilde{\alpha}_B)[\tilde{\beta}_A \cdot \tilde{\beta}_B] \right]_{(S,T)},
\]

Where \([\tilde{\alpha}_A \cdot \tilde{\alpha}_B]_{T}(x) = T(\tilde{\alpha}_A(x), \tilde{\alpha}_B(x)) \) and \([\tilde{\beta}_A \cdot \tilde{\beta}_B]_{S}(x) = S(\tilde{\beta}_A(x), \tilde{\beta}_B(x)) \) for all \( x \in X \).

**Theorem 4.12.** If \( A = (\tilde{\alpha}_A, \tilde{\beta}_A) \) and \( B = (\tilde{\alpha}_B, \tilde{\beta}_B) \) are imaginable interval valued intuitionistic (S, T)-fuzzy Hv-submodules of M. If \( T^* \) (resp. \( S^* \)) is a t-norm (resp. s-norm) which dominates \( T \) (resp. \( S \)), that is, \( T^*(T(\alpha, \beta), T(\gamma, \delta)) \geq T(T^*(\alpha, \gamma), T^*(\beta, \delta)) \) and \( S^*(S(\alpha, \beta), S(\gamma, \delta)) \leq S(S^*(\alpha, \gamma), S^*(\beta, \delta)) \) for all \( \alpha, \beta, \gamma, \delta \in [0,1] \). Then for the intuitionistic \((S^*, T^*)\)-product of A and B, \([A \cdot B]_{(S^*,T^*)}\) is an intuitionistic (S, T)-fuzzy Hv-submodule of M.

**Proof.** In proving this theorem, we only need to verify that the conditions (1)-(4) hold. The verification is mentioned and we omit the details.

Let \( f : M \to M' \) be a strong epimorphism of Hv-modules. Let \( T \) (resp. \( S \)) and \( T^* \) (resp. \( S^* \)) be the t-norms (resp. s-norms) such that \( T^* \) (resp. \( S^* \)) dominates \( T \) (resp. \( S \)). If \( A = (\tilde{\alpha}_A, \tilde{\beta}_A) \) and \( B = (\tilde{\alpha}_B, \tilde{\beta}_B) \) are imaginable interval valued intuitionistic fuzzy Hv-submodules of \( M' \), then the intuitionistic \((S^*, T^*)\)-product of A and B, we have \([A \cdot B]_{(S^*,T^*)}\) is an intuitionistic (S, T)-fuzzy Hv-submodule of \( M' \). Since every strong epimorphism of an intuitionistic (S, T)-fuzzy Hv-submodule is an intuitionistic (S, T)-fuzzy Hv-submodule, the inverse images \( f^{-1}(A), f^{-1}(B) \), and \( f^{-1}([A \cdot B]_{(S^*,T^*)}) \) are also intuitionistic (S, T)-fuzzy Hv-submodules of M. In the next theorem, we described that the relation between \( f^{-1}([A \cdot B]_{(S^*,T^*)}) \) and intuitionistic \((S^*, T^*)\)-product \([f^{-1}(A) \cdot f^{-1}(B)]_{(S^*,T^*)}\) of \( f^{-1}(A) \) and \( f^{-1}(B) \).

Based on the above discussion, we have:

**Theorem 4.13.** Let \( f : M \to M' \) be a strong epimorphism of Hv-modules. Let \( T^* \) (resp. \( S^* \)) be a t-norm (resp. s-norm) such that \( T^* \) (resp. \( S^* \)) dominates \( T \) (resp. \( S \)). If \( A = (\tilde{\alpha}_A, \tilde{\beta}_A) \) and \( B = (\tilde{\alpha}_B, \tilde{\beta}_B) \) are intuitionistic (S, T)-fuzzy Hv-submodule of \( M' \). Then for the intuitionistic \((S^*, T^*)\)-product \([A \cdot B]_{(S^*,T^*)}\) of A and B and the intuitionistic \((S^*, T^*)\)-product \([f^{-1}(A) \cdot f^{-1}(B)]_{(S^*,T^*)}\) of \( f^{-1}(A) \) and \( f^{-1}(B) \) we have \( f^{-1}([A \cdot B]_{(S^*,T^*)}) = [f^{-1}(A) \cdot f^{-1}(B)]_{(S^*,T^*)} \).

**References**


