RESULT ON COMMON FIXED POINT IN FUZZY METRIC SPACE FOR FOUR COMPATIBLE MAPPINGS

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Abstract: The present paper set up a common fixed point theorem in a complete fuzzy metric space for four compatible mappings.

Keywords and Phrases: R-Weakly Commuting Pairs, Compatible Mapping, Self Mappings, Complete Fuzzy Metric Space, Common Fixed Point.


1. INTRODUCTION

Zadeh [30] was a professor emeritus of mathematician with computer scientist, electrical engineer artificial intelligence researcher and best known for proposing fuzzy mathematics consisting of these fuzzy related concepts. Kramosil and Michalek [11] had put forwarded the contraction principal for fuzzy metric space. In general subrahmanyam [25,26] forwarded with generalized, established the result of Garabić [6] for a pair of commuting mappings. R-Weakly commutativity of mappings in fuzzy metric spaces was defined by Vasuki [27]. In this progress lot of researchers namely namely George and Veeramani[4,5], Fuller [3], Gregori and Sapena [7], Imdad, Ali and Hasan [8], Mihet [14], Sastry, Naidu and Krishn [17], Schweizer [18], Bratney and Odeh [13], Romaguera, Sapena and Tirado [16], Shirude and Aage [21], Steimann [24], Vijayaraju and Sajath [28], Singh and Jain [22], Jungck [9], Amari and Moutawakil [1], Mujahid Abbas [2], Sedghi, et.al.[19], Khan [10], Pathak, Lopez and Verma [15], Shen, et.al.[20], Wairojiana, et.al. [29] Recently Soni and Shukla [23] established some fixed point theorem in fuzzy metric space for expansion mapping and proved under different conditions. Manthena and Manchala [12] proved
two common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric space using property E.A.

2. PRELIMINARIES

For this purpose we need the following definitions and Lemmas.

**Definition 2.1.** The 3-tuple \((X, M, \ast)\) is called a fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous t-norm and \(M\) is a fuzzy set on \(X^2 \times (0, \infty)\) satisfying the following conditions:

1. \(M(x, y, t) > 0,\)
2. \(M(x, y, t) = 1\) if and only if \(x = y,\)
3. \(M(x, y, t) = M(y, x, t),\)
4. \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s),\)
5. \(M(x, y, \cdot): (0, \infty) \rightarrow [0, 1]\) is continuous, for all \(x, y, z \in X\) and \(t, s > 0.\)

**Definition 2.2.** A Sequence \(\{x_n\}\) in a fuzzy metric space \((X, M, \ast)\) is a Cauchy sequence if and only if for each \(\varepsilon > 0, t > 0,\) there exists \(n_0 \in \mathbb{N}\) such that \(M(x_n, x_m, t) > 1 - \varepsilon\) for all \(n, m \geq n_0.\)

**Definition 2.3.** Fuzzy metric space \((X, M, \ast)\) is said to be complete if every Cauchy sequence in \((X, M, \ast)\) is a convergent sequence.

A sequence \(\{x_n\}\) in \((X, M, \ast)\) is convergent to \(x \in X\) if

\[
\lim_{n \to \infty} M(x_n, x, t) = 1 \text{ for each } t > 0.
\]

**Definition 2.4.** Two mappings \(f\) and \(g\) on a Fuzzy metric space \((X, M, \ast)\) into itself are said to be Weakly Commuting if

\[
M(fg x, gf x, t) \geq M(f x, g x, t) \text{ for each } x \in X.
\]

**Definition 2.5.** The mapping \(f\) and \(g\) of a Fuzzy metric space \((X, M, \ast)\) into itself are said to be \(R\)-weakly commuting, provided there exists some positive real numbers \(R\) such that

\[
M(fg x, gf x, t) \geq M\left(f x, g x, \frac{t}{R}\right) \text{ for each } x \in X.
\]

**Definition 2.6.** Self mappings \(F\) and \(G\) of a Fuzzy metric space \((X, M, \ast)\) are said to be compatible if and only if \(M(FG x_n, GF x_n, t) \to 1\) for all \(t > 0,\) whenever \(\{x_n\}\) is a sequence in \(X\) such that \(Fx_n, Gx_n \to y,\) for some \(y \in X.\)

**Definition 2.7.** Let \(A\) and \(S\) be self mappings of a Fuzzy metric space \((X, M, \ast)\). We will call \(A\) and \(S\) to be reciprocally continuous if

\[
\lim_{n \to \infty} A S x_n = A p \text{ and } \lim_{n \to \infty} S A x_n = S p \text{ whenever } \{x_n\} \text{ is a sequence such that } \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = p \text{ for some } p \in X.
\]
If A and S are continuous then they are obviously reciprocally continuous. But the converse need not be true.

**Lemma 2.1.** $M(x, y, *)$ is non-decreasing for all $x, y$ in $X$.

**Lemma 2.2.** Let $\{y_n\}$ be a sequence in a Fuzzy metric space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that $M(x_{n+2}, x_{n+1}, k_t) \geq M(x_{n+1}, x_n, t) \forall t > 0, n \in N$, then $\{x_n\}$ is Cauchy sequence in $X$.

### 3 MAIN RESULTS

**Theorem 3.1.** Let $A_1, A_2, A_3$ and $A_4$ be self maps of a complete Fuzzy metric space $(X, M, *)$ with continuous t-norm * defined by

$$a \ast b = \min\{a, b\}, a, b \in [0, 1]$$

(a) $A_1(x) \subset A_4(x), A_2(x) \subset A_3(x),$

(b) $[A_1, A_3]$ or $[A_2, A_4]$ is compatible pair of reciprocally continuous maps,

(c) $[A_1, A_3], [A_2, A_4]$ are point wise $R$-weakly commuting pairs of maps,

(d) For all $x, y \in X, t \in (0, 1), t > 0$

$$M^2(A_1x, A_2y, lt)$$

$$\geq \emptyset \{\min[M^2(A_3x, A_4y, t), M^2(A_1x, A_3y, t), [M^2(A_2y, A_4y, t)$$

$$+ M^2(A_1x, A_4y, t)], [M(A_1x, A_4y, 2t) + M(A_2y, A_3x, 2t)]\}$$

where,

$$\emptyset(t) : R^+ \rightarrow R^+ \text{ is non-decreasing continuous from the right with } \emptyset(t) \geq t \text{ for } t > 0 \text{ and } \emptyset(1) = 1,$$

$$\lim_{t \rightarrow \infty} M(x, y, t) \rightarrow 1.$$

Then $A_1, A_2, A_3$ and $A_4$ have a unique common fixed point in $X$.

**Proof:** Let $x_0 \in X$ be an arbitrarily point in $X$, construct a sequence $\{z_n\}$ in $X$, such that

$$z_{2n-1} = A_4z_{2n-1} = A_1z_{2n-2} \text{ and } z_{2n} = A_3z_{2n} = A_2z_{2n-1} \quad n = 1, 2, 3 \ldots$$

Now,

$$M^2(z_{2n+1}, z_{2n+2}, lt) = M^2(A_1x_{2n}, A_2x_{2n+1}, lt)$$

$$\geq \emptyset \{\min[M^2(A_3x_{2n}, A_4x_{2n+1}, lt), M^2(A_1x_{2n}, A_3x_{2n}, t), [M^2(A_2x_{2n+1}, A_4x_{2n+1}, t)$$

$$+ M^2(A_1x_{2n}, A_4x_{2n+1}, t)], [M(A_1x_{2n}, A_4x_{2n+1}, 2t) + M(A_2x_{2n+1}, A_3x_{2n}, 2t)]\}$$

$$\geq \emptyset \{\min[M^2(z_{2n}, z_{2n+1}, t), M^2(z_{2n}, z_{2n+1}, t), [M^2(z_{2n+2}, z_{2n+1}, t) + M^2(z_{2n+1}, z_{2n+1}, t)]$$

$$+ M(z_{2n+2}, z_{2n+2}, t)]]\}$$

$$\geq \emptyset \{\min[M^2(z_{2n}, z_{2n+1}, t), [M^2(z_{2n+2}, z_{2n+1}, t) + 1], [1 + M(z_{2n}, z_{2n+2}, 2t)]\}$$
If \( M(z_{2n+1}, z_{2n+2}, t) + 1 \leq M(z_{2n}, z_{2n+1}, t) \), then we get a contradiction, and so
\[
M^2(z_{2n+1}, z_{2n+2}, lt) \geq \emptyset [M^2(z_{2n}, z_{2n+1}, t)]
\]
\[
M(z_{2n+1}, z_{2n+2}, lt) \geq M(z_{2n}, z_{2n+1}, t), t > 0 \quad \ldots \quad (a)
\]
Further using (d), we have
\[
M^2(z_{2n}, z_{2n+1}, lt) = M^2(A_2x_{2n-1}, A_1x_{2n}, lt)
\]
\[
= M^2(A_1x_{2n}, A_2x_{2n-1}, lt)
\]
\[
\geq \emptyset \{ \min \{ M^2(A_3x_{2n}, A_4x_{2n-1}, t), M^2(A_1x_{2n}, A_3x_{2n}, t), [ M^2(A_2x_{2n-1}, A_4x_{2n-1}, t) \\
+ M^2(A_1x_{2n}, A_4x_{2n-1}, t), [ M(A_1x_{2n}, A_4x_{2n-1}, 2t) + M(A_2x_{2n-1}, A_3x_{2n}, 2t) ] \} \}
\]
\[
\geq \emptyset \{ \min \{ M^2(y_{2n}, y_{2n+1}, t), M^2(y_{2n}, y_{2n+1}, t), [ M^2(y_{2n}, y_{2n+1}, t) + M^2(y_{2n+1}, y_{2n+1}, t) ] \}
\]
\[
\geq \emptyset \{ \min \{ M^2(y_{2n}, y_{2n+1}, t), M^2(y_{2n}, y_{2n+1}, t), [ M^2(y_{2n}, y_{2n+1}, t) + 1 ] \}
\]
If \( M(y_{2n}, y_{2n+1}, t) + 1 \leq M(y_{2n-1}, y_{2n}, t) \) then we get a contradiction and so
\[
M(y_{2n}, y_{2n+1}, lt) \geq \emptyset [M^2(y_{2n-1}, y_{2n}, t)] \quad \ldots \quad (b)
\]
\[
M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n-1}, y_{2n}, t)
\]
Using (a) and (b), we have
\[
M(y_n, y_{n+1}, lt) \geq M(y_{n-1}, y_n, t) \forall t > 0
\]
This implies that \( \{ y_n \} \) is a Cauchy sequence by Lemma 2.2.

Since \( (X, M, *) \) is complete, so \( \{ y_n \} \) converges to some point \( z \) in \( X \). Thus subsequences \( \{ A_1, x_{2n} \}, \{ A_3, x_{2n} \}, \{ A_2, x_{2n} \} \) and \( \{ A_4, x_{2n} \} \) also converges to \( z \). Suppose \( [A_1, A_3] \) is compatible pair of reciprocally continuous maps. Then by the definition of reciprocally continuous maps, \( A_1, A_3x_{2n} \to A_1z \) and \( A_3, A_1x_{2n} \to A_3z \) and then the compatibility of \( A_1 \) and \( A_3 \) yields \( \lim_{n \to \infty} M(A_1A_3x_{2n}, A_3A_1x_{2n}, t) = 1 \)

i.e. \( M(A_1z, A_3z, t) = 1 \). Hence, \( A_1z = A_3z \)

Since, \( A_1(X) \subset A_4(X) \), there exists a point \( u \) in \( X \) such that \( A_1z = A_4u \).

Using (iv) we have.
\[
M^2(A_1z, A_2u, lt) \geq \emptyset \{ \min \{ M^2(A_3z, A_4u, t), M^2(A_1z, A_3z, t), [ M^2(A_2u, A_4u, t) + M^2(A_1z, A_4u, t) ], [ M(A_1z, A_4u, 2t) + M(A_2u, A_3z, 2t) ] \} \}
\]
\[
\geq \emptyset \{ \min \{ M^2(A_1z, A_1z, t), M^2(A_1z, A_1z, t), [ M^2(A_2u, A_1z, t) + M^2(A_3z, A_1z, t) ], [ M(A_1z, A_1z, 2t) + M(A_2u, A_1z, 2t) ] \} \}
\]
\[ M(A_1 A_3 z, A_3 A_1 z, t) \geq M\left( A_1 z, A_3 z, \frac{l}{R} \right) = 1 \]

i.e. \( A_1 A_3 z = A_3 A_1 z \)

and \( A_1 A_1 z = A_1 A_3 z = A_3 A_1 z = A_3 A_3 z \)

Similarly \( R \)-weakly commutativity of \( A_2 \) and \( A_4 \) implies that

\[ A_2 A_2 u = A_2 A_4 u = A_4 A_2 u = A_4 A_4 u \]

Now by (iv), we have

\[ M^2(A_1 A_1 z, A_1 z, lt) = M^2(A_1 A_1 z, A_2 u, lt) \]

\[ M^2(A_1 A_1 z, A_2 u, lt) \geq \Phi\{M^2(A_2 A_1 z, A_4 u, t), M^2(A_1 A_1 z, A_3 A_1 z, t), [M^2(A_2 u, A_4 u, t) + M^2(A_1 A_1 z, A_4 u, t)], [M(A_1 A_1 z, A_4 u, 2t) + M(A_2 u, A_3 A_1 z, 2t)]\} \]

\[ \geq \Phi\{M^2(A_1 A_1 z, A_1 z, t), M^2(A_1 A_1 z, A_1 A_1 z, t), [M^2(A_1 z, A_1 z, t) + M^2(A_1 A_1 z, A_1 z, t)], [M(A_1 A_1 z, A_1 z, 2t) + M(A_1 z, A_1 A_1 z, 2t)]\} \]

or \( M^2(A_1 A_1 z, A_1 z, lt) \geq \Phi[M^2(A_1 A_1 z, A_1 z, t)] \geq M^2(A_1 A_1 z, A_1 z, t) \)

\[ A_1 A_1 z = A_1 z \]

Thus, \( A_1 z = A_1 A_1 z = A_3 A_1 z \). Thus \( A_1 z \) is a common fixed point of \( A_1 \) and \( A_3 \).

Again by (iv), we have

\[ M^2(A_1 z, A_2 A_2 u, lt) \]

\[ \geq \Phi\{M^2(A_2 z, A_4 A_2 u, t), M^2(A_1 z, A_3 z, t), [M^2(A_2 A_2 u, A_4 A_2 u, t) + M^2(A_1 z, A_4 A_2 u, t)], [M(A_1 z, A_4 A_2 u, 2t) + M(A_2 A_2 u, A_3 z, 2t)]\} \]

\[ \geq \Phi\{M^2(A_1 z, A_2 A_2 u, t), M^2(A_1 z, A_1 z, t), [M^2(A_2 A_2 u, A_2 A_2 u, t) + M^2(A_1 z, A_2 A_2 u, t)], [M(A_1 z, A_2 A_2 u, 2t) + M(A_2 A_2 u, A_1 z, 2t)]\} \]

or \( M^2(A_1 z, A_2 A_2 u, lt) \geq \Phi[M^2(A_1 z, A_2 A_2 u, t)] \geq M^2(A_1 z, A_2 A_2 u, t) \)

This implies that
A_1 z = A_2 A_2 u. Thus, A_2 A_2 u = A_1 z = A_4 A_2 u = A_2 u

Thus, A_2 u (= A_1 z) is a common fixed point of A_2 and A_4 and hence A_1 z is a common fixed point of A_1, A_2, A_3 and A_4.

To prove uniqueness, let A_1 z_1 be another common fixed point of A_1, A_2, A_3 and A_4. Then

\[ M(A_1 z, A_1 z_1, lt) \geq M^2(A_1 A_1 z, A_2 A_1 z_1, lt) \]

\[ \geq \emptyset [\min \{ M^2(A_3 A_1 z, A_4 A_1 z_1, t), M^2(A_1 A_1 z, A_3 A_1 z, t), M^2(A_2 A_1 z_1, A_4 A_1 z_1, t) \}

\[ + M^2(A_1 A_1 z, A_4 A_1 z_1, t), [M(A_1 A_1 z, A_4 A_1 z_1, 2t) + M(A_2 A_1 z_1, A_3 A_1 z, 2t)]]] \]

\[ \geq \emptyset [\min \{ M^2(A_1 z, A_1 z_1, t), M^2(A_1 z, A_1 z_1, t), [M^2(A_1 z_1, A_1 z_1, t) + M^2(A_1 z, A_1 z_1, t)], M(A_1 z, A_1 z_1, 2t) \]

\[ + M(A_1 z_1, A_1 z, 2t)]]] \]

Or \[ M^2(A_1 z, A_1 z_1, lt) \geq \emptyset [M^2(A_1 z, A_1 z_1, t) \geq M^2(A_1 z, A_1 z_1, t) \]

and so \[ A_1 z = A_1 z_1 \]

Thus \[ A_1 z_1 \] is a unique common fixed point of \[ A_1, A_2, A_3 and A_4 \].

REFERENCES


