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A NOTE ON FIXED POINT THEOREMS IN FUZZY METRIC SPACE USING CONTRACTIVE CONDITION

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Abstract: In this paper we have extended the result of fixed point theorem for contractive mappings in fuzzy matric space using control function.

Keywords and Phrases: Control Function, Contractive Condition, Complete Fuzzy Metric Space , Fixed Point Theorem, Integral Type, Rational Expression.

AMS (2010) Subject Classifications: Primary 54H25, Secondary 47H10.

1. INTRODUCTION

Zadeh (1965) introduced the concept of fuzzy sets and till then it has been developed, expanded by many authors in different fields. This theory has wide selection of applications in diverse areas. The strong points about fuzzy mathematics are its fruitful applications, especially outside mathematics, such as in quantum particle physics studied by E l Naschie (2004).

In this paper we extended and generalized the result of Gupta et.al (2015), Grabeic (1988) and also some other result of literature such as Vasuki (1988) , Gregori and Sapena (2002), Gupta and Mani (2014a) and Manthena and Manchala (2018), Gupta,et.al.(2015) .

2.PRELIMINARIES.

Definition 2.1 (Schweizer, 1960) . A binary operation *: [0,1] x [0,1] → [0,1] is a continuous triangular norm (t-norm) if for all a,b,c,e ∈ [0,1] the following conditions are satisfied:

(i) * is commutative and associate,

(ii) a * 1 =a,

(iii) * is continous, and

(iv) $a * b \leq c * e$ whenever $a \leq c$ and $b \leq e$.

A fuzzy metric space in the sense of Kramosil and Michalek (1975) is defined as follows:

Definition 2.2 (Kramosil & Michalek, 1975) The triplet (X, M, *) is said to be fuzzy metric space if X is an arbitrary set , * is continuous t-norm , and M is fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) M(x,y,0) = 0,

(ii) M(x,y,t) = 1, ∀ t > 0 iff x = y,

(iii) M(x, y, t) = M(y, x, t),

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ ∀ x, y, z ∈ X and t, s > 0,

(v) M(x, y, .) : [0, ∞) → [0, 1] is left continuous, and

(vi) $\lim_{t\to\infty} M(x,y,t) = 1$ ∀ x, y ∈ X .

The triplet M(x, y, t) can be taken as the degree of nearness between x and y with respect to

$t \geq 0$.

LEMMA 2.1 *For every x, y ∈ X, the mapping M(x, y, .) is non-decreasing on (0, ∞).*

Grabiec (1988) extended the fixed point theorem of Banach (1922) to fuzzy metric space in sense of Kramosil and Michalek (1975).

THEOREM 2.1 (Grabiec, 1988) *Let (X, M, *) be a complete fuzzy metric space satisfying*

(i) $\lim_{t\to\infty} M\ (x, y, t) = 1$, and

(ii) M(Fx, Fy, kt) $\geq$ M(x, y, t), $\forall$ x, y $\in$ X,

where 0 < k < 1. Then F has a unique fixed point.

Then Vasuki (1988) generalized Grabiecs result for common fixed point theorem for n sequence of mapping in a fuzzy metric space. Gregori and Sapena(2002) gave fixed point theorems for complete fuzzy metric space in the sense of George and Veeraman(1994) and also for Kramosil and Michaleks(1975) fuzzy metric space which are complete in Grabeics sense.

George and Veeramani(1994) modified the concept of fuzzy metric space introduced by Kramosil and Michalek(1975) with the help of t-norm and gave the following definition.

Definition 2. 3 (George & Veeramani, 1994) The triplet (X, M, *) is said to be fuzzy metric space if X is an arbitrary set, * is continuous t-norm, and M is fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) M(x,y,0) > 0,

(ii) M(x,y,t) = 1, $\forall$ t > 0 iff x = y,

(iii) M(x, y, t) = M(y, x, t),

(iv) M(x, y, t) * M(y, z, s) $\leq$ M(x, z, t+s) $\forall$ x, y, z $\in$ X and t, s > 0, and

(v) M(x, y, .) : $[0, \infty) \rightarrow [0, 1]$ is continuous.

By introducing this definition, they also succeeded in introducing a Hausdorff topology on such fuzzy metric spaces which is widely used these days by researchers in their respective field of research. George and Veeramani(1994) have pointed out that the definition of Cauchy sequence given by Grabeic is weaker and hence it is essential to modify that definition to get better results in fuzzy metric space.

Consequently, some more metric fixed point results were generalized to fuzzy metric spaces by various authors such as Subrahmanyam (1995), Vasuki (1998), Saini, Gupta, and Singh (2007), Saini, Kumar, Gupta, and Singh (2008), Vijayaraju (2009), and Gupta and Mani (2014a, 2014b).

Now we give some important definitions and lemmas that are used in sequel.

Definition 2.4 (Grabiec, 1988) A sequence $\{X_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to x $\in$ X if $\lim_{t\to\infty} M(X_n, x, t) = 1\ \forall\ t > 0$.

Definition 2. 5 (Grabiec, 1988) A sequence $\{X_n\}$ in a fuzzy metric space (X, M, *) is called Cauchy Sequence if $\lim_{n\to\infty} (X_{n+p}, X_n, t) = 1 \ \forall\ t > 0$ and each $p > 0$.

Definition 2.6 (Grabiec, 1988) A fuzzy metric space (X, M, *) is said to be complete if every Cauchy sequence in X converges in X.

EXAMPLE 2.1 (Gregori et al., 2011) Let (X, d) be a bounded metric space with $d(x, y) < k$ for all $x, y \in X$. Let $g : R^+ \to (k, \infty)$ be an increasing continuous function. Define a function M as

$$M(x, y, t) = 1 - \frac{d(x,y)}{g(t)},$$

Then (X, M, *) is a fuzzy metric space on X where * is a Lukasievicz t-norm, i.e, $*(a, b) = \max\{a+b-1, 0\}$.

LEMMA 2.2 If there exists $k \in (0,1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all $x, y \in X$ and $t \in (0, \infty)$, then x=y.

In our result, we define a class Φ of all mappings $\xi : [0,1] \to [0,1]$ satisfying the following conditions:

(i) ξ is increasing on [0,1], and

(ii) $\xi(t) > t,\ \forall\ t \in (0, 1]$ and $\xi(t) = t$ if and only if $t = 1$.

In section 3, we prove some fixed point theorems for contractive mappings in fuzzy metric spaces. We prove our results in fuzzy metric spaces in the sense of George and Veeramani (1994). Our result generalizes some relevant results in the literature.

3. MAIN RESULTS

THEOREM 3.1 Let (X, M, *) be a complete fuzzy metric space and $p : X \to X$ be a mapping satisfying

$$M(px, py, kt) \geq \xi\{\lambda(x, y, t)\} \qquad \text{--- (3.1)}$$

Where,

$$\lambda(x, y, t) = Min\left\{M(x, y, t), M(x, px, t), M(y, py, t), \frac{M(x,px,t) * M(y,py,t)}{M(x,y,t)}\right\} \qquad \text{--- (3.2)}$$

for all $x, y \in X$, $\xi \in \phi$ and $k \in (0, 1)$. Then p has a unique fixed point.

PROOF: Let $x \in X$ be any arbitrary point in X. Now construct a sequence $\{X_n\} \in X$ such that

$px_n = x_{n+1}$ for all $n \in N$.

Claim: $\{X_n\}$ is a Cauchy sequence.

Taking $x = x_{n-1}$ and $y = x_n$ in equation 3.1, we get

$M(x_n, x_{n+1}, kt) = M(px_{n-1}, px_n, kt) \geq, \xi\{\lambda(x_{n-1}, x_n, t)$

from equation 3. 2, we have

$\lambda(x_{n-1}, x_n, t)$

$$= Min\left\{M(x_{n-1}, x_n, t), M(x_{n-1}, px_{n-1}, t), M(x_n, px_n, t), \frac{M(x_{n-1}, px_{n-1}, t) + M(x_n, px_n, t)}{M(x_{n-1}, x_n, t)}\right\}$$

$$---(3.3)$$

$$= Min\left\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), \frac{M(x_{n-1}, x_n, t) + M(x_n, x_{n+1}, t)}{M(x_{n-1}, x_n, t)}\right\}$$

$$= Min\{M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\}$$

Now if $M(x_n, x_{n+1}, t) \leq M(x_{n-1}, x_n, t)$ then by equation 3.3,

$M(x_n, x_{n+1}, kt) \geq \xi\{M(x_n, x_{n+1}, t)\} > M(x_n, x_{n+1}, t)$

Hence, our claim follows immediately from Lemma 2..2, Now suppose

$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$, then again from equation 3.3,

$M(x_n, x_{n+1}, kt) \geq \xi\{M(x_{n-1}, x_n, t)\} > M(x_{n-1}, x_n, t)$

Now by simple induction, for all n and $t > 0$, we get

$$M(x_n, x_{n+1}, kt) \geq M\left(x, x_1, \frac{t}{k^{n-1}}\right) \quad ---(3.4)$$

Now for any positive integer r, we have

$$M(x_n, x_{n+r}, t) \geq M\left(x_n, x_{n+1}, \frac{t}{r}\right) * \ldots\ldots\ldots * M\left(x_{n+p-1}, x_{n+p}, \frac{t}{r}\right)$$

Using equation 3.4, we get

$$M(x_n, x_{n+r}, t) \geq M\left(x, x_1, \frac{t}{r\,k^n}\right) *---.* M\left(x, x, \frac{t}{r\,k^n}\right)$$

Taking $\lim_{n\to\infty}$, we get $\lim_{n\to\infty} M(x_n, x_{n+r}, t) = 1$ --- (3.5)

This implies $\{x_n\}$ is a Cauchy Sequence, therefore, there exists a point u ∈ X such that

$$\lim_{n\to\infty} x_n = u$$

CLAIM: u is a fixed point of p.Consider

$$M(u, pu, t) \geq M(px_n, pu, t) * M(u, x_{n+1}, t) \geq \xi \{ \lambda (x_n, u, \tfrac{t}{2k})\} * M(u, x_{n+1}, t) \qquad \text{---(3.6)}$$

Again from Equation 3.2,

$$\lambda (x_n, u, \tfrac{t}{2k}) =$$

$$Min \left\{ M\left(x_n, u, \frac{t}{2k}\right), M\left(x_n, px_n, \frac{t}{2k}\right), M\left(u, pu, \frac{t}{2k}\right), \frac{M\left(x_n, px_n, \frac{t}{2k}\right) * M\left(u, pu, \frac{t}{2k}\right)}{\left(x_n, u, \frac{t}{2k}\right)} \right\}$$

Taking $\lim_{n\to\infty}$, we get

$$\lambda (u, u, \tfrac{t}{2k}) =$$

$$Min \left\{ M\left(u, u, \frac{t}{2k}\right), M\left(u, pu, \frac{t}{2k}\right), M\left(u, pu, \frac{t}{2k}\right), \frac{M\left(u, pu, \frac{t}{2k}\right) * M\left(u, pu, \frac{t}{2k}\right)}{\left(u, u, \frac{t}{2k}\right)} \right\}$$

$$= Min \left\{ 1, \ M\left(u, pu, \frac{t}{2k}\right), M\left(u, pu, \frac{t}{2k}\right), \left(M\left(u, pu, \frac{t}{2k}\right)\right)^2 \right\}$$

$$= M\left(u, pu, \frac{t}{2k}\right)$$

By equation 3.6, we get

$$M(u, pu, t) \geq \xi \left\{ M\left(u, pu, \tfrac{t}{2k}\right) \right\} * M(x_{n+1}, u, t) > M\left(u, pu, \tfrac{t}{2k}\right) * M(x_{n+1}, u, t) \qquad \text{--- (3.7)}$$

Taking $\lim_{n\to\infty}$ in Equation 3.7 and by Lemma 2.2, we get pu = u.

UNIQUENESS: Now we show that u is a unique fixed point of p. Suppose not, then there exists a point z ∈ X such that pz = z.

Consider $1 \geq M(z, u, t) = M(pz, pu, t) \geq \xi\left\{\lambda\left(z, u, \frac{t}{k}\right)\right\}$ ---(3.8)

Where $\lambda\left(z, u, \frac{t}{k}\right) =$

$$Min\left\{M\left(z, u, \frac{t}{k}\right), M\left(z, pz, \frac{t}{k}\right), M\left(u, pu, \frac{t}{k}\right), \frac{M\left(z, pz, \frac{t}{k}\right) * M\left(u, pu, \frac{t}{k}\right)}{M\left(z, u, \frac{t}{k}\right)}\right\}$$

$$= Min\left\{M\left(z, u, \frac{t}{k}\right), M\left(z, z, \frac{t}{k}\right), M\left(u, u, \frac{t}{k}\right), \frac{M\left(z, z, \frac{t}{k}\right) * M\left(u, u, \frac{t}{k}\right)}{M\left(z, u, \frac{t}{k}\right)}\right\}$$

This implies that either $\lambda\left(z, u, \frac{t}{k}\right) = 1$ or $\lambda\left(z, u, \frac{t}{k}\right) = M\left(z, u, \frac{t}{k}\right)$

Using it in equation 3.8, we get z = u.

Thus, u is a unique fixed point of p. This completes the proof of Theorem 3.1.

COROLLARY 3.1. Let (X, M, *) be a complete fuzzy metric space and p :X → X be a mapping satisfying

$M(px, py, kt) \geq \lambda(x, y, t)$

where,

$$\lambda(x, y, t) = Min\left\{M(x, y, t), M(x, px, t), M(y, py, t), \frac{M(x, px, t) * M(y, py, t)}{M(x, y, t)}\right\}$$

for all x, y ∈ X, and k ∈ (0, 1). Then p has a unique fixed point . The proof of the result follows immediately from Theorem 3.1 by taking ξ(t) = t.

4.APPLICATION.

THEOREM 4.1

In this section, we give an application related to our result.

Let us define $\psi : [0, \infty] \to [0, \infty]$, as

$\Psi(t) = \int_0^t \Psi(t)dt \ \ \forall\, t > 0$, be a non-decreasing and continuous function. Moreover, for each $\epsilon > 0$, $\Psi(t) > 0$ and $\Psi(t) = 0$ iff $t = 0$.

Let (X, M, *) be a complete fuzzy metric space and $p : X \to X$ be a mapping satisfying

$$\int_0^{M(px,py,kt)} \Psi(t)\, dt > \xi \left\{ \int_0^{\lambda\,(x,y,t)} \Psi(t)\ dt \right\}$$

Where,

$\lambda\,(x, y, t) =$

$$Min\left\{ M(x,y,t), M(x,px,t), M(y,py,t), \frac{M(x,px,t) * M(y,\ py,t)}{M(x,y,\ t)} \right\}$$

for all $x, y \in X$, $\phi \in \Psi$, $\xi \in \emptyset$ and $k \in (0, 1)$, Then p has a unique fixed point.

Proof: By taking $\Psi(t) = 1$ and applying Theorem 3.1, we obtain the result.

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