Some Crown of Wheel related Path union families of product cordial graphs

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***Abstract*: We show that for crown graphs W_4^+,W_5^+,W_6^+ path union of families are product cordial graphs. Further we show that W_n^+ is product cordial.**

***Keywords*: labeling, cordial, product, wheel, crown.**

Subject Classification: 05C78

2. Introduction: The graphs we consider are simple, finite, undirected and connected. For terminology and definitions we depend on Graph Theory by Harary [6], A dynamic survey of graph labeling by J.Gallian [7] and Douglas West.[8]. I.Cahit introduced the concept of cordial labeling [5].There are variety of cordial labeling available in labeling of graphs. Sundaram, Ponraj, and Somasundaram [9] introduced the notion of product cordial labeling. A product cordial labeling of a graph G with vertex set V is a function f from V to {0, 1} such that if each edge uv is assigned the label f(u)f(v), the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph. We use $v_f(0,1) = (a, b)$ to denote the number of vertices with label 1 are a in number and the number of vertices with label 0 are b in number. Similar notion on edges follows for $e_f(0,1) = (x, y)$.
A lot of work is done in this type of labeling so far. One interested in survey may refer Dynamic survey in Graph labeling by J. Gallian.We mention a very short part of it. Sundaram, Ponraj, and Somasundaram have shown that trees; unicyclic graphs of odd order; triangular snakes; dragons; helms; Pm∪Pn; Cm∪Pn; Pm∪K1,n; Wm∪Fn (Fn is the fan Pn+K1); K1,m∪K1,n; Wm∪ K1,n; Wm ∪Pn; Wm ∪Cn; the total graph of Pn (the total graph of Pn has vertex set V (Pn)∪E(Pn) with two vertices adjacent whenever they are neighbors in Pn); Cn if and only if n is odd; $C_n^{(t)}$, the one-point union of t copies of C_n, provided t is even or both t and n are even; K2+mK1 if and only if m is odd; $C_m∪P_n$ if and only if m+n is odd; $K_{m,n}$ ∪Ps if s > mn; Cn+2∪K1,n; Kn∪Kn,(n−1)/2 when n is odd; Kn∪Kn−1,n/2 when n is even; and P2 n if and only if n is odd. They also prove that $K_{m,n}$ (m,n > 2), $P_m \times P_n$ (m,n > 2) and wheels are not product cordial and if a (p,q)-graph is product cordial graph, then q 6 (p−1)(p + 1)/4 + 1. In this paper We show that path union of W_4^+,W_5^+,W_6^+ are families of product cordial graphs. Further we show that w_n^+ is also product cordial.

3. Preliminaries:

3.1 Fusion of vertex. Let G be a (p,q) graph. Let u≠v be two vertices of G. We replace them with single vertex w and all edges incident with u and that with v are made incident with w. If a loop is formed is deleted. The new graph has p-1vertices and at least q-1 edges.If u϶G_1 and v϶G_2, where G_1 is (p_1,q_1) and G_2 is (p_2,q_2) graph. Take a new vertex w and all the edges incident to u and v are joined to w and vertices u and v are deleted. The new graph has p_1+p_2-1 vertices and $q_1 + q_2$ edges. Sometimes this is referred as u is identified with the concept is well elaborated in D. West [9]

3.2 **Path union of G** i.e. $P_m(G)$ is obtained by taking a path P_m and m copies of graph G. Fuse a copy each of G at every vertex of path at given fixed point on G. It has mp vertices and mq +m-1 edges, where G is a (p, q) graph. If we change the vertex on G that is fused with vertex of P_m then we generally get a path union non isomorphic to earlier structure. In this paper we define a product cordial function f that does not depends on which vertex of given graph G is used to obtain path union. This allows us to obtain path union in which the same graph G is fused with vertices of P_m at different vertices of G, as our choice and the same function f is applicable to all such structures that are possible on $P_m(G)$.

3.3 Crown graph. It is C_n □K_2.At each vertex of cycle a n edge was attached. We develop the concept further to obtain crown for any graph. Thus crown (G) is a graph G □K_2.It has a pendent edge attached to each of it's vertex. If G is a (p,q) graph then crown(G) has q+p edges and 2p vertices.

4. Main Results:

4.1 Theorem: $G = P_m(w_4^+)$, the Path union of w_4^+ is product cordial.

Proof: To define $P_m(w_4^+)$ take a path $P_m= (v_1,e_1, v_2, e_2, .., e_{m-1}, v_m)$. At each vertex of it fuse a copy of W_4^+ at given fixed vertex. The ordinary labeling of i^{th} copy of w_4^+ fused at i^{th} vertex of P_m is given by hub is w_i, the cycle vertices $u_{i,1}$, $u_{i,2}$,.. $u_{i,4}$,cycle edges are $c_{i,j} = (u_{i,j}u_{i,j+1})$ where j = 1, 2,3,4. and j+1 taken (modulo 4).The pendent vertices are $w_{i,1}$, $w_{i,2}$, $w_{i,3}$, $w_{i,4}$, $w_{i,5}$ with $w_{i,j}$ adjacent to $u_{i,j}$ by edge $(u_{i,j}w_{i,j})$ for i = 1,2, 3, 4. And $w_{i,5}$ adjacent to w_i by an edge $(w_iw_{i,5})$.

Define a function f : V(G) → {0,1} given by:

Using f we get two type of labeling on w_4^+ Type A and Type B namely.Further On path Om label each vertex as ‘1’.

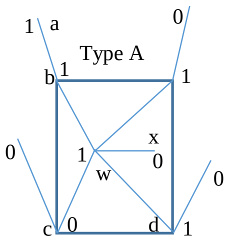

Fig. 4.1 : w_4^+: vertex labels are shown.$e_f(0,1) = (7,6)$; $v_f(0,1) = (5,5)$

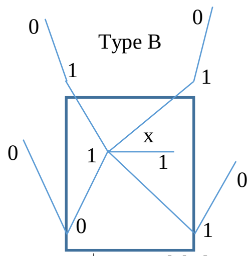

Fig. 4.2 : w_4^+: vertex labels are shown.$e_f(0,1) = (7,6)$; $v_f(0,1) = (5,5)$

Fuse a copy of w_4^+ (type A label) at each vertex of P_m .The point of fusion is vertex ‘a’ on w_4^+.(refer fig.4.1).

The label number distribution is $v_f(0,1) = (5m, 5m)$ and $e_f(0,1) = (7m, 7m-1)$.

It is important to note that the same function f as above will produce product cordial but structurally different (up to isomorphism) path union resulting in same label distribution if we take path union at point ‘d’ or ‘w’ as shown in fig 4.1. If we have to take path union at point ‘x’ we will use Type B labeling instead of Type A label in construction of path union. The function f and the label distribution will remain same.

Thus the graph is product cordial. #

4.2 Theorem. $G = P_m(w_5^+)$, the Path union of w_5^+ is product cordial.

Proof: To define $P_m(w_5^+)$ take a path $P_m= (v_1,e_1, v_2, e_2, .., e_{m-1}, v_m)$. At each vertex of it fuse a copy of w_4^+ at given fixed vertex. The ordinary labeling of i^{th} copy of w_5^+ fused at i^{th} vertex of P_m is given by hub is w_i, the cycle vertices $u_{i,1}, u_{i,2},.. u_{i,5}$ And cycle edges are $c_{i,j} =(u_{i,j}u_{i,j+1})$ where j = 1, 2,,..5. and j+1 taken (modulo 5).The pendent vertices are $w_{i,1}$, $w_{i,2}$, $w_{i,3}$, $w_{i,4}$, $w_{i,5}$,$w_{i,6}$ with $w_{i,j}$ adjacent to $u_{i,j}$ by edge $(u_{i,j}w_{i,j})$ for i = 1,2, 3, 4,5 And $w_{i,6}$ adjacent to w_i by an edge $(w_iw_{i,6})$. From type X label it follows that we can design four types of path unions on W_5^+ at points ‘a’, ‘c’ ‘w’ and ‘y’.

It is not possible to define product cordial labeling on path union at vertex w, the hub of w_5^+.

Define a function $f : V(G) \rightarrow \{0,1\}$ given by:

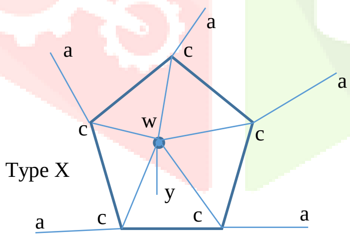

Fig. 4.3 $v_f(0,1) = (6,6)$, $e_f(0,1) = (8,8)$

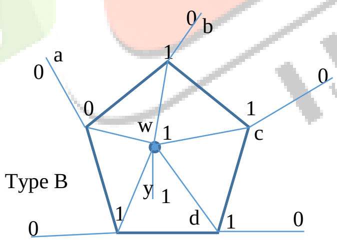

Fig. 4.4 $v_f(0,1) = (6,6)$, $e_f(0,1) = (8,8)$

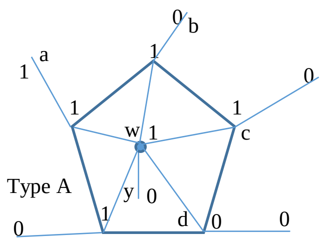

Fig. 4.5 $v_f(0,1) = (6,6)$, $e_f(0,1) = (8,8)$

We get three types of labels on W_5^+ as Type A, type B, type C as above. The Path vertices are labeled as: f(v1) = f(v2) = 1;f(vi)=0 for i≡ 3(mod 4); $f(v_i) = 1$ for i≡ 0,1,2(mod 4). (i ≠1, 2)

Structure 1: To take path union at pendent vertex we use type A label at pendent vertex ‘a’ on it and fuse it at path vertex v_i when i≡ 0,1,2 (mod 4) further vertex ‘b’ on type A label is used if i ≡ 3(mod 4). Structure 2: Path union is taken at point c (fig. 4.3) of W_5^+ . we use type A label vertex ‘c’ on it and fuse it at path vertex v_i when i≡ 0,1,2 (mod 4) further vertex ‘d’ on type A label is used if i ≡ 3(mod 4). Structure 3: Path union is taken at point y (fig. 4.3) of W_5^+ . we use type B label at ‘y’ on it and fuse it at path vertex v_i when i≡ 0,1,2 (mod 4) further vertex 'y' on type A label is used if i ≡ 3(mod 4). In all structures label number distribution is given by $v_f(0,1)$ = (6m,6m),$e_f(0,1)$= (8m+x, 8m+x) ;m=2x+1 where m is an odd number. For even m we have $v_f(0,1)$ = (6m,6m),$e_f(0,1)$= (8m+x-1, 8m+x); m = 2x 4.3 Theorem G = $P_m(w_6^+)$, the Path union of w_6^+ is product cordial.

To define $P_m(w_6^+)$ take a path P_m= ($v_1,e_1, v_2, e_2, .., e_{m-1}, v_m$). At each vertex of it fuse a copy of w_6^+ at given fixed vertex. The ordinary labeling of i^{th} copy of w_6^+ fused at i^{th} vertex of P_m is given by hub is w_i, the cycle vertices $u_{i,1}, u_{i,2},..$ $u_{i,5},u_{i,6}$ And cycle edges are $c_{i,j} =(u_{i,j}u_{i,j+1})$ where j = 1, 2,,..6. and j+1 taken (modulo 6).The pendent vertices are $w_{i,1}, w_{i,2}, w_{i,3}, w_{i,4}, w_{i,5}, w_{i,6}, w_{i,7}$ with w_i adjacent to $u_{i,j}$ by edge $(u_{i,j}w_j)$ for i = 1,2, ,,,m, j = 1, 2,..6 And $w_{i,7}$ is adjacent to w_i by an edge $(w_iw_{i,7})$. From fig. 4.5 it follows that we can design four types of path unions on W_6^+ by taking fusion with path vertex v_i at points 'a', 'b' 'c' and 'd' of W_6^+. Define a function f : V(G) → {0,1} given by: For path $f(v_i)$ = 1 for all i = 1, 2, ..,m. For labeling of i^{th} copy follow the diagram 4.5. Since the labeling of W_6^+ is independent of path labels. While fusing W_6^+ at vertex v_i of P_m care should be taken that fusing vertex label on copy of W_6^+ is '1'. In structure 1 we fuse W_6^+ at vertex x on it with every vertex of path Pm. In structure 2 we fuse W_6^+ at vertex y on it with every vertex of path Pm. In structure 3 we fuse W_6^+ at vertex c or hub vertex w_i on it with every vertex of path Pm. In structure 4 we fuse W_6^+ at vertex d on it with every vertex of path Pm. In all cases we get the label number distribution is given by $v_f(0,1)$ = (7m,7m),$e_f(0,1)$= (10m, 10m-1)4.4 4.4 Theorem W_n^+ is product cordial. Proof: We define W_n^+ as follows. Take cycle C_n given by (v_1, e_1, v_2, e_2, .., e_{m-1}, v_n, e_n, v_1). Take a vertex w not on C_n. Take edges $c_i = (wv_j)$; j = 1, 2, ..,n .Take new n vertices given by $u_1, u_2, ..u_n$ and u_{n+1}. Take edges $b_i=(v_iu_i)$; i = 1, 2, ..n. and (wu_{n+1}).Thus the graph G= W_n^+ has 2n +2 vertices and 3n+1edges. Define a function f : V(G) → {0,1} given by: case 1 n = 2x+ 1, x = 1, 2, ..

f(w) =1 ;

$f(v_i)$ = 1 for i = 1, 2, ..,x+2;

$f(v_i)$ = 0 for i = x+3, x+4, ..n.

$f(u_i)$ = 1 for i =1, 2, ..,(x-1) ,

f(uj)= 0 for j= x, x+1, ..n;

$f(u_{n+1})$= 0;

The label number distribution is given by $v_f(0,1)$ =(2x+2, 2x+2), $e_f(0,1)$ = (3x+2,3x+2), Case n = 2x, x= 2, 3, 4, ..

f(w) =1 ;

$f(v_i)$ = 1 for i = 1, 2, ..,x+1;

$f(v_i)$ = 0 for i = x+2, x+3, ..n.

$f(u_i)$ = 1 for i =1, 2, ..,(x-1) ,

f(u)= 0 for j= x, x+1, ..n;

$f(u_{n+1})$= 0;

The label number distribution is given by $v_f(0,1)$ =(2x+1, 2x+1); $e_f(0,1)$ = (3x,3x+1).

Conclusions: In this paper we have defined Crown graph for C_n given by C_n^+ , crown of W_n given by W_n^+.We have shown that path union of W_4^+,W_5^+,W_6^+ are families of product cordial graphs. We have also obtained particular product cordial labels of C_n^+ and that of W_n^+.In discussion on path union we obtain different structures and it depends which point on G is used to obtain path union. The product cordial function f we define gives particular product cordial label numbers for all such structure except for $P_m(w_5^+)$ where if path union is taken on hub, the product cordial labeling is not available. And this remains a challenge to meet in future.

Future Scope: We have to obtain product cordial labeling for ($P_m(w_n^+)$, $P_m(w_8^{++})$, , .. $P_m(w_n^{+t})$). Where $P_m(w_n^{+t})$ is obtained by fusing t pendent edges with each vertex w_n.

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