The Role of Laplace Transformation In Digital Signal Processing

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Abstract: **In this work we used Laplace transformation in digital signal processing. The main purpose of this paper is to demonstrate how Laplace transformation techniques can be useful in signal processing, convolution, Fourier analysis. It teaches that a linear system can be completely understood from its impulse or frequency response and with increasing complexity of engineering problems. This is a very generalized approach. In fact, it is too general for many applications in science and engineering. Many of the parameters in our universe interact through differential equation. In this paper will discuss the Laplace transformation is a technique for analysing the special systems when the signals are continuous.**

Index Terms - Laplace transformation, Fourier transformation, s-domain, convolution.

I. Introduction

In mathematics, the Laplace transform is a widely used integral transform. It has many important applications in mathematics, physics, engineering and probability theory. The Laplace transform is related to the Fourier transform, but whereas the Fourier transformer solves a function or signal into its modes of vibration, the Laplace transform resolves a function into. Like the Fourier transform, the Laplace transform is used for solving differential and integral equations. In physics and engineering, it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. In this analysis, the Laplace transform is often interpreted as a transformation from the time domain, in which inputs and outputs are functions of time, to the frequency-domain, where the same inputs and outputs are functions of complex angular frequency, in radians per unit time. Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional often simplifies the process of analyzing the behaviour of the system, or in synthesizing a new system based on a set of specifications. Denoted L{x(t)}, it is a Linear operator on a function x (t) (original) with a real argument ($t \geq 0$) that transforms it to a function X(s) (image) with a complex argument s. This transformation is essentially bijective for the majority of practical uses; the respective pairs of x(t) and X (s) are matched in tables. The Laplace transform has the useful property that many relationships and operations over the originals x(t) correspond to simpler relationships and operations over the images X(s).

1.1 History

The Laplace transform is a well established mathematical technique for solving differential equations. It is named in honour of the great French mathematician, Pierre Simon De Laplace (1749-1827). Like all transforms, the Laplace transform changes one signal into another according to some fixed set of rules or equations. A digital signal processing, receives an electrical signal of microsecond duration, for example, representing the data, extracts the data from the signal, sends it to the output, and then goes back to repeat the process. This is the general nature of our technology today. Although many of our engineering systems run over the infinite time, like global positioning system receivers in our cell phones, traffic light signalling systems at the street intersections etc., but if we examine the internal implementation details we will find that the repeating processes of finite time are everywhere.

II. The Nature of The S-Domain

The Laplace transform changes a signal in the time domain into a signal in the s-domain, also called the s-plane. The time domain signal is continuous, extends to both positive and negative infinity, and may be either periodic or a periodic. The Laplace transform allows the time domain to be complex; however, this is seldom needed in signal processing. In this discussion, and nearly all practical applications, the time domain signal is completely real. The s-domain is a complex plane, i.e., there are real numbers along the horizontal axis and imaginary numbers along the vertical axis. The distance along the real axis is expressed by the variable, σ, Likewise; the imaginary axis uses the variable, ω, the natural frequency. This coordinate system allows the location of any point to be specified by providing values for σ and ω. Using complex notation, each location is represented by the complex variable, s, where: $s = \sigma + j\omega$. Just as with the Fourier transform, signals in the s-domain are represented by capital letters. For example, a time domain signal, x (t), is transformed into an s-domain signal, X(s), or alternatively, X (σ, ω). The s-plane is continuous, and extends to infinity in all four directions.

In addition to having a location defined by a complex number, each point in the s-domain has a value that is a complex number. In other words, each location in the s-plane has a real part and an imaginary part. As with all complex numbers, the real & imaginary parts can alternatively be expressed as the magnitude & phase.

2.1 Proof Of The Laplace Transformation In Digital Signal Processing

Just as the Fourier transform analyzes signals in terms of sinusoids, the Laplace transform analyzes signals in terms of sinusoids and exponentials. From a mathematical standpoint, this makes the Fourier transform a subset of the more elaborate Laplace transform. Figure -1 shows a graphical description of how the s-domain is related to the time domain. To find the values along a vertical line in the s-plane (the values at a particular σ), the time domain signal is first multiplied by the exponential curve: $e^{-\sigma t}$. The left half of the s-plane & multiplies the time domain with exponentials that increase with time (σ < 0), while in the right half the exponentials decrease with time (σ > 0). Next, take the complex Fourier transform of the exponentially weighted signal. The resulting spectrum is placed along a vertical line in the s-plane, with the top half of the s-plane containing the positive frequencies and the bottom half containing the negative frequencies. Take special note that the values on the y-axis of the s-plane (σ = 0) are exactly equal to the Fourier transform of the time domain signal.

The complex Fourier Transform is given by:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

This can be expanded into the Laplace transform by first multiplying the time domain signal by the exponential term:

$$X(\sigma,\omega) = \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t}\, dt$$

While this is not the simplest form of the Laplace transform, it is probably the best description of the strategy and operation of the technique.
STEP 1: Start with the time domain signal called x(t).
STEP 2: Multiply the time domain signal by an infinite number of exponential curves, each with a different decay constant, σ. That is, calculate the signal: x (t)$e^{-\sigma t}$ for each value of σ from negative to positive infinity.
STEP 3: Take the complex Fourier Transform of each exponentially weighted time domain signal. to calculate:

$$\int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t}\, dt$$

For each value of σ from negative to positive infinity.
STEP 4: Arrange each spectrum along a vertical line in the s-plane. The positive frequencies are in the upper half of the s-plane while the negative frequencies are in the lower half.

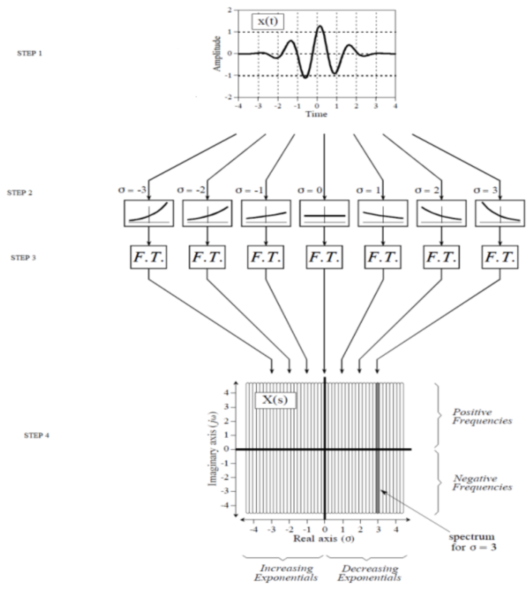

Figure-1

The Laplace transform converts a signal in the *time domain*, *x* (*t*), into a signal in the *s-domain* X(*s*) or *X* (σ, ω). The values along each vertical line in the s-domain can be found by multiplying the time domain signal by an exponential curve with a decay constant σ, and taking the complex Fourier transform. When the time domain is entirely real, the upper half of the s-plane is a mirror image of the lower half.

The two exponential terms can be combined:

$$X(\sigma,\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

Finally, the *location* in the complex plane can be represented by the complex variable, *s*, where *s*=σ+jω. This allows the equation to be reduced to an even more compact expression:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \qquad \text{-------(1)}$$

This equation defines how a time domain signal, *x* (*t*), is related to an s-domain signal, *X*(*s*). The s-domain variables, *s*, and *X*( ) , are complex. While the time domain *may* be complex, it is usually real. This is the final form of the Laplace transform, one of the most important equations in signal processing and electronics. Pay special attention to the term: e^{-st} called a *complex exponential*. As shown by the above derivation, complex exponentials are a compact way of representing both sinusoids and exponentials in a single expression.

Although we have explained the Laplace transform as a two stage process (multiplication by an exponential curve followed by the Fourier transform), keep in mind that this is only a teaching aid, a way of breaking Eq-1 into simpler components. The Laplace transform is a single equation relating *x* (*t*) and *X*(*s*), not a step-by-step procedure. Equation-1 describes how to calculate each *point* in the s-plane (identified by its values for σ and ω) based on the values of σ, ω, and the time domain signal, *x* (*t*). Using the Fourier transform to *simultaneously* calculate all the points along a vertical line is merely a convenience, not a requirement. However, it is very important to remember that the values in the s-plane along the y-axis (σ = 0) are *exactly* equal to the Fourier transform. This is a key part of why the Laplace transform is useful.

To explore the nature of Eq-1 further, let's look at several individual points in the s-domain and examine how the values at these locations are related to the time domain signal. To start, recall how individual points in the *frequency domain* are related to the time domain signal. Each point in the frequency domain, identified by a specific value of ω, corresponds to two sinusoids, cos(ω*t*) and sin(ω*t*). The real part is found by multiplying the time domain signal by the cosine wave, and then integrating from -∞ to∞. The imaginary part is found in the same way, except the sine wave is used. If we are dealing with the *complex* Fourier transform, the values at the corresponding negative frequency, -ω, will be the complex conjugate (same real part, negative imaginary part) of the values at ω. The Laplace transform is just an extension of these same concepts.

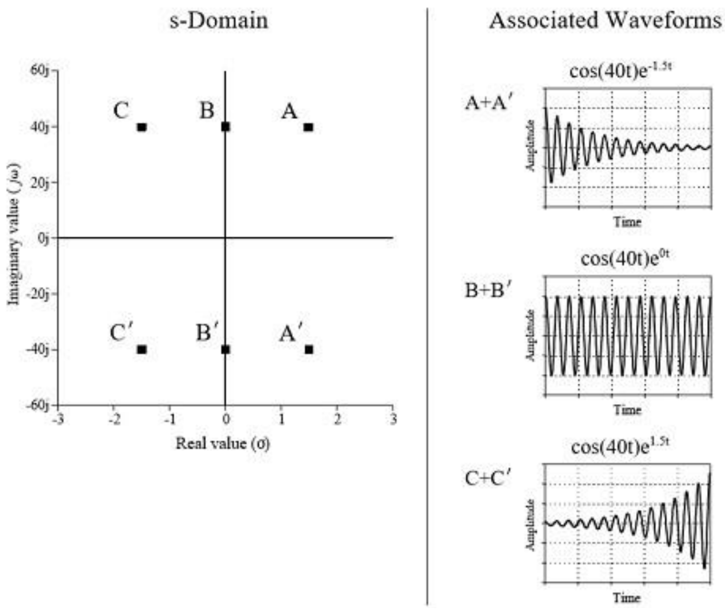

Figure -2: Waveforms associated with the s-domain. Each location in the s-domain is identified by two parameters: σ and ω. These parameters also define two waveforms associated with each location. If we only consider pairs of points (such as: A&A[l], B&B[l], and C&C[l]), the two waveforms associated with each location are sine and cosine waves of frequency ω, with an exponentially changing amplitude controlled by σ.

Figure-2 shows three pairs of points in the s-plane: A&A[l], B&B[l], and C&C[l]. Just as in the complex frequency spectrum, the points at A, B, & C (the positive frequencies) are the complex conjugates of the points at A[l], B[l], & C[l] (the negative frequencies). The top half of the s-plane is a mirror image of the lower half, and both halves are needed to correspond with a real time domain signal. In other words, treating these points in pairs bypasses the complex math, allowing us to operate in the time domain with only real numbers.

Since each of these pairs has specific values for σ and ±ω, there are two waveforms associated with each pair: and . For cos(ωt) $e^{-\sigma t}$ & sin(ωt) $e^{-\sigma t}$ instance, points A&A[l] are at a location of σ = 1.5 and ω = ±40 and therefore correspond to the waveforms: cos(40t) $e^{-1.5t}$ & sin(40t) $e^{-1.5t}$. As shown in Fig.-2, these are sinusoid that exponentially decreases in amplitude as time progresses. In this same way, the sine and cosine waves associated with B&B[l] have constant amplitude, resulting from the value of σ being zero. Likewise, the sine and cosine waves that are associated with locations C&C[l] exponentially increases in amplitude, since σ is negative.

The value at each location in the s-plane consists of a real part and an imaginary part. The real part is found by multiplying the time domain signal by the exponentially weighted cosine wave and then integrated from -∞ to ∞. The imaginary part is found in the same way, except the exponentially weighted sine wave is used instead. It looks like this in equation form, using the real part of A&A[l] as an example:

$$\mathrm{Re}X(\sigma = 1.5, \omega = \pm 40) = \int_{-\infty}^{\infty} x(t)\cos(40t)e^{-1.5t}\, dt$$

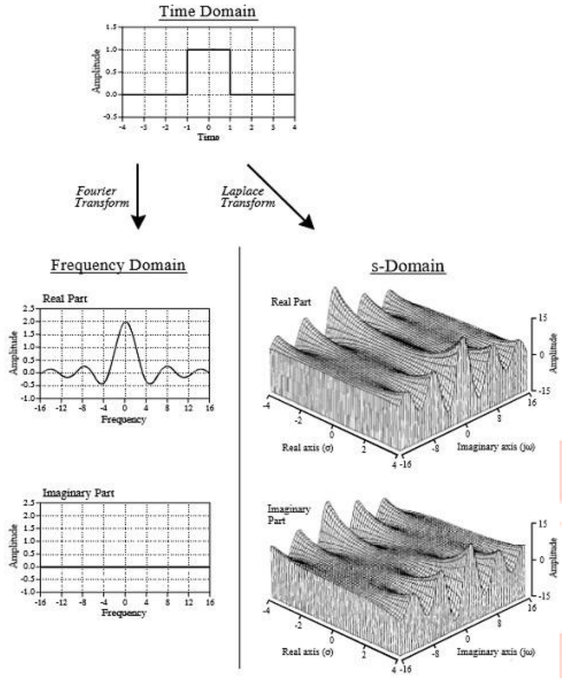

Figure-3: Time, frequency and s-domains. A time domain signal (the rectangular pulse) is transformed into the frequency domain using the Fourier transform, and into the s-domain using the Laplace transform.

Above diagram shows an example of a time domain waveform, its frequency spectrum, and its s-domain representation. The example time domain signal is a rectangular pulse of width two and height one. As shown, the complex Fourier transform of this signal is a sine function in the real part and an entirely zero signal in the imaginary part. The s-domain is an undulating two dimensional signal, displayed here as topographical surfaces of the real and imaginary parts. The mathematics works like this:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\, dt = \int_{-1}^{1} 1e^{-st}\, dt$$

In words, we start with the definition of the Laplace transform (Eq-1), plug in the unity value for x(t), and change the limits to match the length of the nonzero portion of the time domain signal. Evaluating this integral provides the s-domain signal, expressed in terms of the complex location, s, and the complex value X(s):

$$X(s) = \frac{e^{s} - e^{-s}}{s}$$

While this is the most compact form of the answer, the use of complex variables makes it difficult to understand, and impossible to generate a visual display, such as Fig-3. The solution is to replace the complex variable, s, with σ+jω and then separate the real and imaginary parts:

$$\mathrm{Re}X(\sigma,\omega) = \frac{\sigma\cos(\omega)[e^{\sigma} - e^{-\sigma}] + \omega\sin(\omega)[e^{\sigma} + e^{-\sigma}]}{\sigma^2 + \omega^2}$$

$$\mathrm{Img}X(\sigma,\omega) = \frac{\sigma\sin(\omega)[e^{\sigma} + e^{-\sigma}] + \omega\cos(\omega)[e^{\sigma} - e^{-\sigma}]}{\sigma^2 + \omega^2}$$

The topographical surfaces in Fig.-3 are graphs of these equations. These equations are quite long and the mathematics to derive them is very tedious. that these equations reduce to the Fourier transform along the y-axis. This is done by setting σ to zero in the equations, and simplifying:

$$ReX(\sigma,\omega)\Big|_{\sigma=0} = \frac{2\sin(\omega)}{\omega} \qquad ImX(\sigma,\omega)\Big|_{\sigma=0} = 0$$

As illustrated in Fig.-3, these are the correct frequency domain signals, the same as found by directly taking the Fourier transform of the time domain waveform.

III. Conclusions

The paper presented the application of Laplace transform in different areas of physics and electrical power engineering. Besides these, Laplace transform is a very effective mathematical tool and this technique can be useful in signal processing, convolution, Fourier transformation. It becomes an integral part of modern science, being used in a vast number of different disciplines. Whether they are being used in electrical circuit analysis, signal processing, or even in modelling radioactive decay in nuclear physics, they have quickly gained popularity among the intellectual community that deals with these subjects on a day to day basis.

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