PREDICTION OF VARIABLE USING PARAMETRIC FRONTIER PRODUCTION FUNCTION – THEORETICAL APPROACH

[1]Dr N Ramesh Kumar, [2]Dr S Chandra Babu, [3]Smt N Sujana Kumari, [4]Dr S Hari Prasad

[1,2]Lecturer in STATISTICS, [3]Lecturer in MATHEMATICS, [4] Asst Professor in STATISTICS,
Department of STATISTICS,
SV Arts College, Tirupati, India

ABSTRACT: **The present study aims at predicting potential revenue against target cost and potential inputs against target revenue. It distinguishes inefficient decision making units(DMS's) from efficient units. Two parametric frontier production functions (i) the Cobb-Douglas and (ii) the variable returns to scale are considered as basic tools. Their dual factor minimal cost frontiers and expressions for potential input vectors for target revenue and potential revenue for target cost are derived.**

KEYWORDS: **Frontier production function, Decision making units, Variable returns to scale, Cobb-Douglas production function, Input technical efficiency measure, Output technical efficiency measure.**

INTRODUCTION

Most of the parametric frontier production functions handle one output and multi-inputs. Further, if output level is specified and input substitution is possible, infinitely many input combinations prevail such that each input combination is capable of producing the pre-specified output. Thus, with one specified output level we can associate an infinite number of input combinations.

Potential revenues and potential inputs are prediction, in particular, when parametric frontiers are used expressions of factor minimal cost and optimal revenue function are required. These functional forms depend on certain parameters, which are estimated by postulating and solving certain programming problems. Since these estimates lack statistical properties, they are not amenable for any tests of significance.

For prediction Shephard's input and output sets are considered

Input sets :

$$L\left(\frac{r}{R}\right) = \{x : R(x, r) \geq R\} = \left\{x : R\left(x, \frac{r}{R}\right) \geq 1\right\}$$

Where x : *Input vector*

$R(x,r)$: Optimal revenue

r : Out price vector

R : *Target revenue*

OUTPUT SETS:

$$P\left(\frac{p}{c}\right) = \{u : Q(u, p) \leq c\}$$

$$= \left\{u : Q\left(u, \frac{p}{c}\right) \leq 1\right\}$$

Where, u : *Output vector*

p: Input price vector

c : *Target revenue*

Potential revenue is obtained by solving,

$$r\ (p) \quad = \underset{u}{\mathrm{Max}}\{u : Q(u, p)\ \leq\ c\}$$

Potential inputs are obtained by solving,

$$\mathrm{Min}\{\lambda : R(\lambda\, x, r)\ \geq\ R\} = \ \hat{\lambda}$$

Potential efficient inputs : $\hat{\lambda}\, x$

INPUT SETS $\mathbf{L(u)}$– FRONTIER PRODUCTION FUNCTION $\mathbf{f(x)}$:

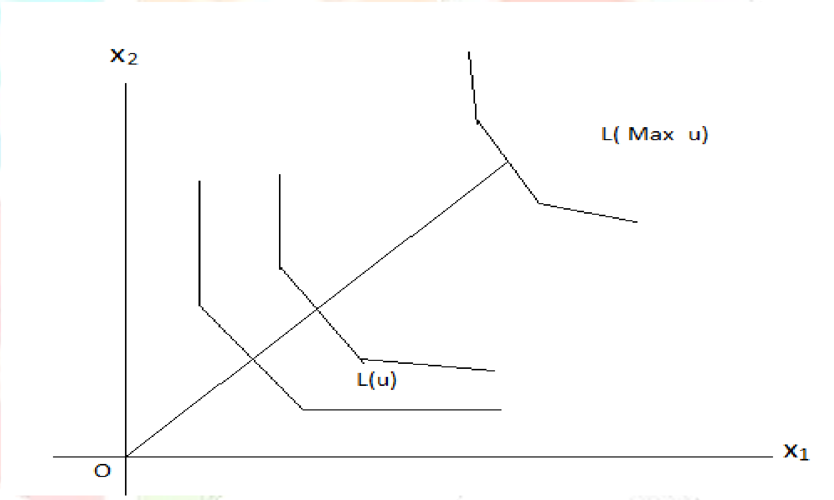
The input sets $\mathbf{L(u)}$ and the frontier production function $\mathbf{f(x)}$ determine each other completely.

Input sets in terms of frontier production function are expressed as,

$$L(u) = \{x : f(x)\ \geq\ u\}$$

The implicit assumption to express input sets in terms of frontier production function is that the output produced is scalar valued.

For a given scalar output, the input set $\mathbf{L(u)}$ constitutes all input vectors capable of producing 'u' or more than 'u'. The frontier production function f(x) can be expressed in terms of input sets as follows.



The production unit that employs the input vector 'x', can produce all outputs consistent with inputs consistent with input sets whose isoquants fall below $Isoq\ \ L(Max\ \ u)$. The maximum output it produces correspond the input set whose isoquant is $Isoq\ \ L(Max\ \ u)$.

THE COBB-DOUGLAS FRONTIER:

If u_0 and x_0 are the observed output and input vectors, we always have,

$$u_0\ \leq\ A \underset{i}{\pi} {x_{io}}^{\alpha_i}$$

The input and output technical efficiency measures are related as follows:

$$\lambda^{\theta} = \delta^{-1}$$

λ: Input technical efficiency measure

θ : *The returns to scale parameter*

δ : *The output technical efficiency measure*

THEVARIABL RETURN TO SCALE FRONTIER:

Suppose u_0 and u^* be observed and frontier outputs. If these are scalar valued the output pure technical efficiency is obtained by solving the following equation.

$$(u^*)^{\propto} e^{\beta u^*} = A \overset{\pi}{\underset{i}{}} x_{io}{}^{\propto_i}$$

In this equation only unknown is u[1]*. The pure output technical efficiency measure is,

$$\delta = \frac{u^*}{u_o} = \frac{\delta u_o}{u_o}$$

Elasticity of scale : $\in = \frac{1}{\propto + \beta u}$

If return to scale are consistent, elasticity of scale equals to unity.

Consequently, the scale efficient output is,

$$u^c = \frac{1-\propto}{\beta}$$

Observed output : u_0

The input overall technical efficiency:

$$\lambda^c = \left(\frac{e\beta}{1-\propto}\right)^{1-\propto} \Big/ A \overset{\pi}{\underset{i}{}} \left(\frac{x_{io}}{u_o}\right)^{\propto_i},$$

The output overall technical efficiency

$$\delta^c = (\lambda^c)^{-1}$$

$$\delta^c = A \overset{\pi}{\underset{i}{}} \left(\frac{x_{io}}{u_o}\right)^{\propto_i} \Big/ \left(\frac{e\beta}{1-\propto}\right)^{1-\propto}$$

VARIABLE RETURNS TO SCALE:

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[1]*Reduce inputs radially in the direction of origin till the following equality is achieved.

$$u_o \leq A \overset{\pi}{\underset{i}{}} (\lambda x_i)^{\alpha_i}$$

$$\lambda^{\sum \alpha_i} A \overset{\pi}{\underset{i}{}} x_i{}^{\alpha_i}$$

$$u_o = \lambda^{\theta} A \overset{\pi}{\underset{i}{}} x_i{}^{\alpha_i} \text{Where } \theta = \sum \alpha_i$$

Output augmentation is such that,

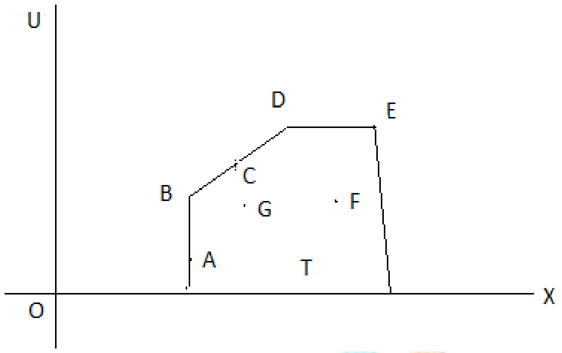
$$\delta u_o = A \overset{\pi}{\underset{i}{}} x_i{}^{\alpha_i}$$

By comparison $\lambda^{\theta} = \delta^{-1}$

The production possibility set,

$$T=\left\{ (x,u) : \sum_{j=1}^{N} \lambda_j u_j \le x\,,\ \sum_{j=1}^{N} \lambda_j u_j \ge u\,,\ \lambda_j \ge 0,\ \sum_{j=1}^{N} \lambda_j \le 1, \right\}$$

Admits variable returns to scale



The production units A,B,C,D and E determine the variable returns to scale frontier. Returns to scale of the DMU's A, C and E are respectively increasing and decreasing.

PRIMAL PROBLEM – RETURNS TO SCALE:

$$\text{Max } Z = \sum_j w_j u_{j0} - \eta$$

Subject to $\sum_j w_j u_{jk} - \sum_j v_i x_{ik} - \eta \le 0$

$$\sum_i v_i x_{ik} = 1,\ \le w_j,\ v_j > 0$$

(i) $\eta > 0 \rightarrow$ increasing return to scale

(ii) $\eta = 0 \rightarrow$ Constant return to scale

(iii) $\eta < 0 \rightarrow$ Decreasing return to scale

ZELLNER – REVANKAR [2]* VARIABLE RETURNS TO SCALE PRODUTION FRONTIER:

$$u\ e^{\theta u} = A \prod_{i=1}^{n} x_i^{\alpha_i},\quad 0 \le \alpha_i \le 1\ ,\quad \sum_i \alpha_i = 1$$

$$f(u)=\emptyset(x)$$

$\emptyset(x)$ is homogeneous of degree one in inputs

$F(u)$ is not homogeneous of degree one in u

The output sets induced by the VRS frontier possess homothetic input structure but not output structure.

VARIABLE RETURNS TO SCALE PRODUTION FRONTIER – MAXIMUM REVENUE:

$$u^{\alpha} e^{\theta u} = A \prod_{i=1}^{n} x_i^{\alpha_i},$$

$\sum_i \alpha_i = 1$ is the variable returns to scale frontier

[2]* Zellner,A.,and Revanker, N.S.(1969), 'Generalized Production Functions, Review of Economic Studies, 36, 241-250.

$$\emptyset(x) = A \prod_{i=1}^{n} x_i^{\alpha_i},$$

$$f(u) = u^{\alpha} e^{\theta u}$$

$f(u)$ is not linear homogeneous in u, consequent to which $R(x, r)$ does not split into the product $\emptyset(x)\, B(r)$.

VARIABLE RETURNS TO SCALE PRODUCTION FRONTIER-MAXIMUM REVENUE

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ESTIMATION OF COBB-DOUGLAS PRODUCTION FRONTIER:

C.P. Timmer proposed a linear programming problem whose solution yields the Cobb-Douglas frontier production function

$$\min_{\Theta \in \otimes} \frac{\emptyset(x_0)}{u_0}$$

$$\text{subject to } \frac{\emptyset(x_j)}{x_j} \geq 1, \quad j = 1, 2, \ldots\ldots, n$$

'x_0' and 'u_0' are the input vector and scalar output of the production unit for which the objective function is formulated. 'x_j' and 'x_j' are the input vector and scalar output of the j[th] production unit in competition. 'Θ' is the parametric vector and $\otimes$ is the parametric space.

$$\min_{\Theta \in \otimes} \frac{\prod_i x_{i0}^{\alpha_i}}{u_0}$$

$$\text{Subject to } \frac{A \prod_i x_{ij}^{\alpha_i}}{u_j} > 1, j = 1, 2, \ldots, n, \quad A > 0 \ and\ 0 \geq \alpha_i \leq 1$$

Upon taking logarithms this non-liner programming problem can be transformed into the following linear programming problem

$$\min_{(A, \alpha_i)} \ln A + \sum_i \alpha_i \ln x_{i0}$$

Subject to $\ln A + \sum_i \alpha_i \ln x_{ij} \geq \ln u_j \quad j = 1, 2, \ldots, n \ and\ 0 < \alpha_i < 1, i = 1, 2, \ldots\ldots, m \quad \ln A$ is unrestricted for sign

For given inputs and outputs of 'n' production units the linear programming problem is solved for the unknown parameters A and α_i, $(i = 1, 2, \ldots, m)$.

Since, there are 'n' production units competing in production, one can solve 'n' linear programming problems not obtaining a single frontier under which all the n production units operates.

In order to obtain a single frontier, in the places of inputs $\ln x_{i0}$ their mean values can be used. Consequently, we solve the following linear programming problem

$$\underset{(\ln A, \propto_i)}{\text{Min}} \ln A + \sum_{i=1}^{m} \propto_i \overline{\ln x_{i0}}$$

Subject to $\ln A + \sum_i \propto_i \ln x_{ij} \geq \ln u_j$

$ln\ \ A$ is unrestricted for sign $0 < \propto_i < 1, i = 1,2, \ldots \ldots, m$

EASTIMATION OF VRS PRODUCTION FRONTIER:

$$\underset{\Theta \in \otimes}{\text{Min}} \frac{\emptyset(x_0)}{f(u_0)}$$

Subject to $\frac{\emptyset(x_j)}{x_j} \geq 1, \quad j = 1,2, \ldots \ldots, n$

$\emptyset(x) = A \prod_{i=1}^{n} x_i^{\propto_i},$

$f(u) = u^{\propto} e^{\theta u}$

$$\underset{\Theta \in \otimes}{\text{Min}} \frac{\prod_i x_{i0}^{\propto_i}}{u_0}$$

Subject to $\frac{A \prod_i x_{ij}^{\propto_i}}{u_j} > 1, \ j = 1,2, \ldots \ldots, n, \quad A > 0, \ \ 0 \geq \propto_i \leq 1$

By using logarithms this problem can be transformed into a linear programming problem

$$\underset{(A, \propto_i \propto, \theta)}{\text{Min}} \ln A + \sum_i \propto_i \ln x_{i0} - \propto \ln u_0 - \theta u_0$$

Subject to $\ln A + \sum_i \propto_i \ln x_{ij} - \propto \ln u_j - \theta u_j \geq 0$

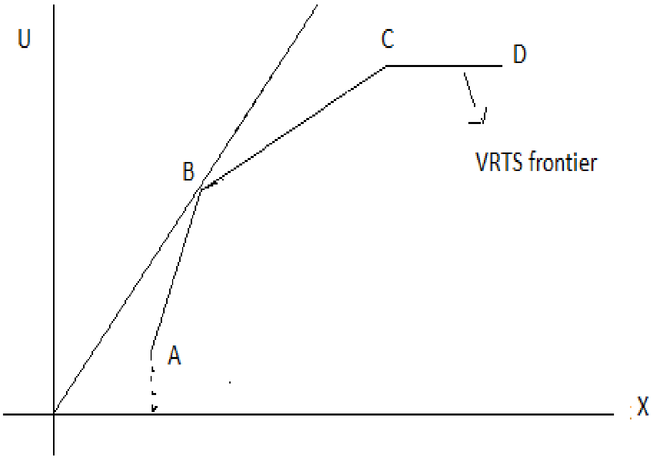
$0 < \propto_i < 1, i = 1,2, \ldots, m$

$ln\ A$ is unrestricted for sign.

To obtain unique frontier under which an the production units operate. We solve the following liner programming problem

$$\left.\begin{array}{l} \underset{(A, \propto_i \propto, \theta)}{\text{Min}} \ln A + \sum_i \propto_i \ln x_{i0} - \propto \ln u_0 - \theta u_0 \\ \text{Subject to} \ln A + \sum_i \propto_i \ln x_{ij} - \propto \ln u_j - \theta u_j \geq 0 \\ 0 < \propto_i < 1, i = 1,2, \ldots, m \\ \text{In A is unrestricted for sign} \end{array}\right\}$$

VARIABLE RETURNS TO SCALE:



The production units operating at A,B,C,D generate a graph that admits variable returns to scale. Returns to scale at A are increasing, B are constant, C and D are decreasing. The production technology is piece-wise linear.

$$GR^V = \left\{ (x, u) : \sum_j \lambda_j X_j \le x, \quad \sum_j \lambda_j u_j \ge u, \quad \lambda_j \ge 0, \sum_j \lambda_j \le 1, \right\}$$

OUTPUT PRODUCTION:

When target cost is specified we wish to estimate the potential output. There are two cases.

(i) Output 'u' is scalar valued

The potential output (cost limited maximal output) is found solving,

$$\underset{u}{\text{Max}} \{u : \ Q(u,p) \le C\}$$

$$\underset{u}{\text{Max}} = \left\{ r\,u : p\,x \le C, \sum_j \lambda_j x_j \le x, \sum_j \lambda_j u_j \ge u, \quad \lambda_j \ge 0, \sum_j \lambda_j \le 1 \right\}$$

For each target cost we have to solve one linear programming problem

(ii) The output 'u' is vector valued. Then we find the potential output by solving the optimization problem,

$$\text{Max}\lambda$$

Such that $Q(\lambda\,\bar{u}, p) \le C$ or

$$\text{Max}\lambda$$

Such that
$$\left. \begin{array}{c} p\,x \le C \\ \sum_j \lambda_j x_j \le \bar{x}, \\ \sum_j \lambda_j u_j \ge u \\ \lambda_j \ge 0 \\ \sum_j \lambda_j = 1, \end{array} \right\}$$

Where $\bar{x}$ is the input vector currently employed by the production unit whose output prediction is under consideration

Potential output $= [\text{Max } \lambda]\ \bar{u}$

INPUT PREDICTION:

When target revenue is specified we wish to estimate the necessary efficient input required to meet the target revenue. There are two cases

(i) Input 'x' is scalar valued the potential input is obtained solving the following optimization problem:

$$\underset{x}{\text{Max}} \{ x : R(x,r) \geq R \}$$

$$\underset{x}{\text{Max}} = \left\{ x : r\,u \geq R,\ \sum_j \lambda_j x_j \leq x,\ \sum_j \lambda_j u_j \geq u,\ \ \lambda_j \geq 0,\ \sum_j \lambda_j = 1, \right\}$$

For each target revenue, we have to solve one linear programming problem.

(ii) The input x is vector valued. The potential input vector is obtained solving the

following linear programming problem

Min λ

Such that $Q(\lambda\, x, r) \geq R$ Or

Min λ

Such that $r\,u \geq R,$

$\sum_j \lambda_j x_j \leq x,$

$\sum_j \lambda_j u_j \geq u,$

$\lambda_j \geq 0,\ \sum_j \lambda_j = 1,$

Potential input $= [\text{Min } \lambda]\bar{x}$

Where '$\bar{x}$' is the input vector currently employed by the production unit whose output prediction is under consideration.

In variable prediction discussed and implemented in this work the chief tool is production function, either parametrically postulated or empirically constructed. Two parametric specifications are considered,

(a) The Cobb-Douglas frontier, which has a simple structure, but very widely used both in empirical and theoretical research and

(b) The variable returns to scale production frontier for which the former frontier is a special case.

CONCLUSION

A producer or policy maker targets revenue and desires to predict potential inputs. He is equally interested to predict potential revenue against a target cost. The present study attempts to suggest methods to predict potential inputs against target revenues and potential revenues against target costs.

One method rests on Cob-Douglas frontier production and cost function and another is based on empirically constructed production frontiers. The estimates obtained have differed only marginally.

Measurement and determination of returns to scale is straight forward in the case of parametric frontiers such as CD frontier. Returnsto scale is a frontier's surface property. If production unit is inefficient it operates below the frontier. Its inefficient output will be projected vertically onto the frontier; consequently returns to scale are measured at this point. Alternatively, its inefficient inputs are reduced radial such that a frontier point is encountered, at which returns to scale are measured.

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