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INERTIA OF DISTANCE MATRIX OF SPIDER GRAPH Distance Matrix

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Abstract :-

Let D denote the distance matrix of a connected graph G. The inertia of D is the triple of integers $(n_+(D), n_-(D), n_0(D))$, where $n_+(D), n_-(D), n_0(D)$ denote the number of positive, negative and 0 eigenvalues of D, respectively. In this paper, we will find the inertia of distance matrix of spider graph which is a extension of wheel graph. [1]

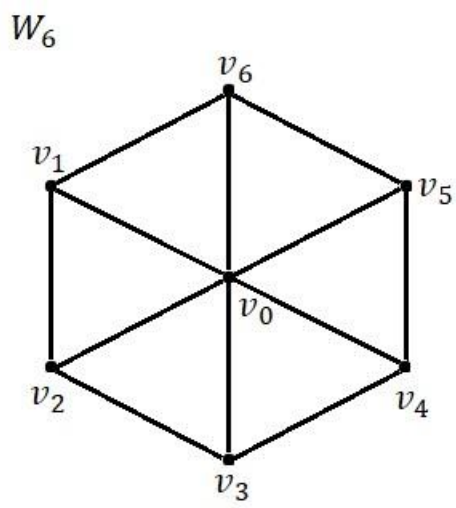
1. Introduction :-

Let G be a undirected connected graph with n vertices. Let $V(G) = \{v_1, v_2, ..., v_n\}$, then the distance between two vertices v_i and v_j is the length of shortest path between v_i and v_j, denoted by $d_G(v_i, v_j)$. The distance matrix of a graph is defined in a similar way as the adjacency matrix: the entry in the i^{th} row, j^{th} th column is the distance between the i^{th} and j^{th} vertex. In this paper we will denote distance matrix of garph G by G only. The D $-eigenvalues$ of a graph G are the eigenvalues of its distance matrix G which form the distance spectrum or $D - spectrum$ of G.

The inertia of a real symmetric matrix G is triple (x,y,z) where x,y,z are the number of positive, negative and zero eigenvalues of distance matrix of a graph G, respectively. It is denoted by $In(G) = (x,y,z)$.

Definition:- WHEEL GRAPH, W_n

The wheel graph on $n + 1$ vertices W_n is a graph that contains a cycle of length n and vertex v_0 (sometimes called the hub) not in the cycle such that v_0 is connected to every other vertex.



Definition:- SPIIDER GRAPH, $W_{n,k}$

The spider graph $(W_{n,k})$ on nk+1 vertices is a graph whose vertices set is $V(W_{n,k}) = \{v_0\} \cup \{v_1^{(j)}, v_2^{(j)}, \cdots, v_n^{(j)} | j =$ 0,1,⋯ ,k − 1} and edges set is $E(W_{n,k}) = \{v_0 v_i^{(0)} | i = 1, 2, \cdots, n\} \cup \{v_i^{(j)} v_i^{(j+1)} | i = 1, 2, \cdots, n; j = 0, 1, \cdots, k -$ 1} ∪ E_r, where $E_r = \{v_1^{(r)} v_2^{(r)}, v_2^{(r)} v_3^{(r)}, \cdots, v_{n-1}^{(r)} v_n^{(r)}, v_n^{(r)} v_1^{(r)}\}$, for $r = 0, 1, \cdots, k - 1$.

Note:

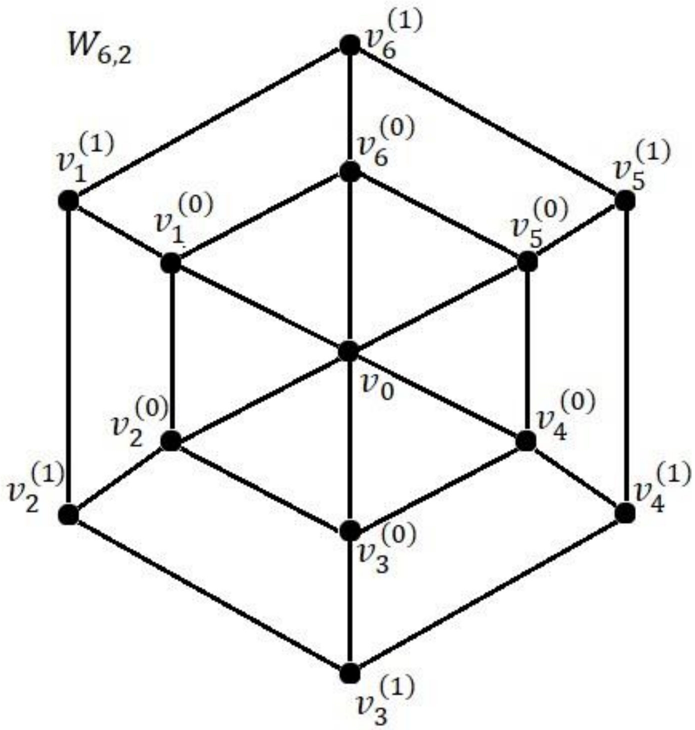
1) **$W_{n,1}$ is wheel on $n + 1$ vertices.**
2) **E_r forms a cycle of length n.**

2) $|V(W_{n,k})| = nk + 1$ **3)** $|E(W_{n,k})| = 2nk$.

Constuction of Spider Graph:

1) **Draw a wheel on** $n + 1$ **vertices lablled by center** v_0 **and other vertices by** $v_1^{(0)}, v_2^{(0)}, \cdots, v_n^{(0)}$.
2) **Draw a cycle** $C_n^{(1)} : v_1^{(1)} - v_2^{(1)} - \cdots - v_n^{(1)} - v_1^{(1)}$ **around wheel.**
3) **Add edges** $v_i^{(0)} v_i^{(1)}$, **for** $i = 1,2,\cdots,n$. **4) Continue above upto** k **cycles.**

e.g. $W_{4,2}$**:**



Cauchy's Interlacing Theorem

Let A **be a Hermitian matrix of order** n **and** B **be a principal submatrix of** A **of order** $n - 1$. **If** $\lambda_n \le \lambda_{n-1} \le \lambda_{n-2} \le \cdots \le \lambda_2 \le \lambda_1$ **lists the eigenvalues of** A **and** $\mu_n \le \mu_{n-1} \le \mu_{n-2} \le \cdots \le \mu_3 \le \mu_2$ **lists the eigenvalues of** B. **Then** $\lambda_n \le \mu_n \le \lambda_{n-1} \le \mu_{n-1} \le \cdots \le \mu_3 \le \lambda_2 \le \mu_2 \le \lambda_1$.**(?,?)**

Theorem 1:-

Let

$$T_n = \begin{vmatrix} -2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & -2 \end{vmatrix}_{n\times n}$$

Then $T_n = (-1)^n(n + 1)$

Proof :- **Expanding the determinant by first row, we get**

$$T_n = (-2)\begin{vmatrix} -2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & -2 \end{vmatrix}_{(n-1)\times(n-1)} - (-1)\begin{vmatrix} -1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & -2 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= (-2)T_{n-1} - T_{n-2}$$

$\therefore T_n + 2T_{n-1} + T_{n-2} = 0$ **and** $T_1 = -2, T_2 = (-2)(-2) - (-1)(-1) = 3$ **We will solve the above recurrence relation.**

Axullary equation is $\alpha^2 + 2\alpha + 1 = 0$

$\therefore (\alpha + 1)^2 = 0 \Longrightarrow \alpha = -1,-1$

General solution is $T_n = (c_1 + nc_2)(-1)^n$.

By using given condition. we get,

$T_1 = (c_1 + (1)c_2)(-1)^1 \Longrightarrow -2 = (c_1 + c_2)(-1) \Longrightarrow 2 = c_1 + c_2$

$T_2 = (c_1 + (2)c_2)(-1)^2 \Longrightarrow 3 = (c_1 + 2c_2)(-1)^2 \Longrightarrow 3 = c_1 + 2c_2$

By solving we get $c_1 = c_2 = 1$

$\therefore T_n = (1 + n)(-1)^n$

Hence proved. ...□□□

Theorem 2:-

Let $\widetilde{W_{n,k}}$ be a Spider Graph , for $n \geq 3$ & $k \geq 1$. Let $D(W_{n,k})$ be the distance matrix of $W_{n,k}$ and $\widehat{D}(W_{n,k})$ be the principal submatrix of $D(W_{n,k})$. Then

1) $n_0(\widehat{D}(W_{n,k})) = (n-1)(k-1)$
2) $n_+(\widehat{D}(W_{n,k})) = 0$
3) $n_-(\widehat{D}(W_{n,k})) = n + k - 1$

Proof :-

We have distance matrix of Spider graph as follow:

$$D(W_{n,k}) = \begin{bmatrix} 0 & L & 2L & \cdots & kL \\ L^t & B_1 & B_2 & \cdots & B_k \\ 2L^t & B_2 & B_1 & \cdots & B_{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ kL^t & B_k & B_{k-1} & \cdots & B_1 \end{bmatrix}_{(nk+1)\times(nk+1)}$$

Where, $L = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}_{1\times n}$, L^t **is a transpose of** L.

$$B_1 = \begin{bmatrix} 0 & 1 & 2 & \cdots & 2 & 1 \\ 1 & 0 & 1 & \cdots & 2 & 2 \\ 2 & 1 & 0 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & 0 & 1 \\ 1 & 2 & 2 & \cdots & 1 & 0 \end{bmatrix}_{n\times n} = D(W_n) \text{ and } B_r = B_1 + (r-1)J_n$$

Where $D(W_n)$ is distance matrix of wheel graph on $n+1$ vertices.

To find principal submatrix $\widetilde{D(W_{n,k})}$, we subtract 1^{st} row from remaining nk rows 1^{st} column from remaining nk column. Removing 1^{st} row and 1^{st} column we get,

$$\widehat{D}(W_{n,k}) = \begin{bmatrix} S_1 & S_1 & S_1 & \cdots & S_1 \\ S_1 & S_2 & S_2 & \cdots & S_2 \\ S_1 & S_2 & S_3 & \cdots & S_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_1 & S_2 & S_3 & \cdots & S_k \end{bmatrix}_{nk\times nk}$$

$$\text{Where } S_1 = \begin{bmatrix} -2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & -2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & -2 \end{bmatrix}_{n} = \widehat{D}(W_n) = B_1 - 2J_n \text{ and } S_r = S_1 - 2(r-1)J_n = B_1 - 2rJ_n$$

Since, $L^tL = J_n$

and $B_r - iL^tL - jL^tL = B_1 + (r-1)J_n - iJ_n - jJ_n = B_1 + (r-i-j-1)J_n$ **For $i \leq j$, we have** $r = j - i + 1$

$\therefore B_r - iL^tL - jL^tL = B_1 + (j - i + 1 - i - j - 1)J_n = B_1 - 2iJ_n = S_i$

Similarlly, for $i \geq j$, $B_r - iL^tL - jL^tL = S_j$

To prove : $n_+(\widehat{D}(W_{n,k})) = 0$

It is sufficient to prove that $\widehat{D}(W_{n,k})$ is negative semi-definite.

We will prove it by minor test.

Note: **Matrix** $A = [a_{ij}]_{n\times n}$ **is said to be negative semi-definite if** $(-1)^iD_i \geq 0$.

Where,

$$D_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{vmatrix}_{i\times i}$$

Consider

$$det(\widetilde{D(W_{n,k})}) = \begin{vmatrix} S_1 & S_1 & S_1 & \cdots & S_1 \\ S_1 & S_2 & S_2 & \cdots & S_2 \\ S_1 & S_2 & S_3 & \cdots & S_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_1 & S_2 & S_3 & \cdots & S_k \end{vmatrix}_{nk\times nk}$$

By subtracting $k-1^{th}$ block row from k^{th} block row, $k-2^{th}$ block row from $k-1^{th}$ block row,

...

1^{st} block row from 2^{nd} block row,

We get,

$$det(\widetilde{D(W_{n,k})}) = \begin{vmatrix} S_1 & S_1 & S_1 & \cdots & S_1 \\ 0 & S_2-S_1 & S_2-S_1 & \cdots & S_2-S_1 \\ 0 & 0 & S_3-S_2 & \cdots & S_3-S_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S_k-S_{k-1} \end{vmatrix}_{nk\times nk}$$

But

$$S_r - S_{r-1} = (B_1 - 2rJ_n) - (B_1 - 2(r-1)J_n) = -2J_n$$

$$\therefore det(\widetilde{D(W_{n,k})}) = \begin{vmatrix} S_1 & S_1 & S_1 & \cdots & S_1 \\ 0 & -2J_n & -2J_n & \cdots & -2J_n \\ 0 & 0 & -2J_n & \cdots & -2J_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2J_n \end{vmatrix}_{nk\times nk}$$

$$= \begin{vmatrix} -2 & -1 & 0 & \cdots & -1 & -2 & -1 & 0 & \cdots & -1\cdots \\ -1 & -2 & -1 & \cdots & 0 & -1 & -2 & -1 & \cdots & 0\cdots \\ 0 & -1 & -2 & \cdots & 0 & 0 & -1 & -2 & \cdots & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\cdots \\ -1 & 0 & 0 & \cdots & -2 & -1 & 0 & 0 & \cdots & -2\cdots \\ 0 & 0 & 0 & \cdots & 0 & -2 & -2 & -2 & \cdots & -2\cdots \\ 0 & 0 & 0 & \cdots & 0 & -2 & -2 & -2 & \cdots & -2\cdots \\ 0 & 0 & 0 & \cdots & 0 & -2 & -2 & -2 & \cdots & -2\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\cdots \\ 0 & 0 & 0 & \cdots & 0 & -2 & -2 & -2 & \cdots & -2\cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\cdots \end{vmatrix}_{nk\times nk}$$

From the above determinant, we get Minors as follow.

T_i $i < n$ $i = n$

$det(S_1)$ $i = n + 1$ i

$D_i =$ $> n + 1$

$-2det(S$

0 $_1)$

We have $T_i = (-1)^i(i+1)$ **&** $det(S1) = C_n = 2((-1)^n - 1)$ **...by Theorem 1**

$(-1)^i(i+1)$ $i < n$

$2((-1)^n - 1)$ $i = n$

$\therefore D^i =$ -0 $4((-1)^n - 1)ii > n= n + 1+ 1$

$(i+1)$ $i < n$

(-1)

$\therefore^i D^i =$ $2(14(10 \;\; -- ((--1)1)^n n))$ $iii > n== nn + 1+ 1$

$\therefore (-1)^i D_i \geq 0$

$\therefore \widetilde{D(W_{n,k})}$^ **is negative semi-definite.**

$\therefore D(W_{n,k})$ **has no positive eigen value.**

$\therefore n_+(D\hat{}(W_{n,k})) = 0$

Also we know that "For a symetric matrix, Nullity of matrix = no of zero eigen values." We can see that each row in 2^{nd} block rows is same.

$\therefore$ it contributes $n-1$ to the nulluty of $\widetilde{D(W_{n,k})}$. Similarlly, 3^{rd} block row, $\cdots$, k^{th} block row contributes $n-1$ to the nulluty of $\widetilde{D(W_{n,k})}$. Remaining all rows are linearly independent. $\therefore Nullity(D\hat{}(W_{n,k})) = (n-1)(k-1)$

$\therefore n_0(D^\wedge(W_{n,k})) = (n-1)(k-1)$

$\therefore n_-(D^\wedge(W_{n,k})) = nk - n_0(D^\wedge(W_{n,k})) - n_+(D^\wedge(W_{n,k})) = nk - (n-1)(k-1) - 0 = n + k - 1$ ...□□

Theorem 3:-

For a Spider Graph $W_{n,k}$**,** $n \geq 3$ **&** $k \geq 1$

1) $n_0(D(W_{n,k})) = (n-1)(k-1)$

2) $n_+(D(W_{n,k})) = 1$

3) $n_-(D(W_{n,k})) = n + k - 1$

Proof :-

Let $D^\wedge(W_{n,k})$ **be the principal submatrix of** $D(W_{n,k})$**.**

We have $D^\wedge(W_{n,k})$ **is negative semidefinite.**

$\therefore$ **eigenvalues of** $\widetilde{D(W_{n,k})}$ **^ are either zero or negative.**

We have $n_0(D(W_{n,k})) = (n-1)(k-1)$ **and** $n_-(\widetilde{D(W_{n,k})}) = n + k - 1$

Let $\mu_1 = \mu_2 = \cdots = \mu_{(n-1)(k-1)} = 0$ **and** $\mu_{(n-1)(k-1)+1}, \cdots, \mu_{nk} < 0$ **be the eigenvalues of** $\widetilde{D(W_{n,k})}$**. Let** $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{nk+1}$ **be the eigenvalues of** $D(W_{n,k})$**.**

$\therefore$ **by Cauchy's interlacing theorem,** $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{(n-1)(k-1)} \geq \mu_{(n-1)(k-1)} \geq \lambda_{(n-1)(k-1)+1} \geq \mu_{(n-1)(k-1)+1} \geq \cdots \geq \mu_{nk} \geq \lambda_{nk+1}$

$\therefore \lambda_1 \geq 0 \geq \lambda_2 \geq 0 \geq \cdots \geq \lambda_{(n-1)(k-1)} \geq 0 \geq \lambda_{(n-1)(k-1)+1} \geq \mu_{(n-1)(k-1)+1} \geq \cdots \geq \mu_{nk} \geq \lambda_{nk+1}$

Since λ_1 **be the only non negative eigenvalue of** $D(W_{n,k})$ **and** $trace\{D(W_{n,k})\} = 0$**, therefore** $\lambda_1 > 0$

$\therefore n_+(D(W_{n,k})) = 1$ **also** $n_-(D(W_{n,k})) = n + k - 1$

$\therefore n_0(D(W_{n,k})) = (nk+1) - (n+k-1) - 1 = (n-1)(k-1)$ ...□□

References

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