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PROJECTIVE CHANGES OF FINSTER METRICS BY AN h-VECTOR

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ABSTRACT

In the present paper we have determined the conditions under which a geodesic of a Finsler space $F^n = (M^n, L)$ is also a geodesic of Finsler space $\bar{F}^n = (M^n, \bar{L})$ and vice versa underlying with the same manifold M^n, where $\bar{L} = f(L, \beta)$ is a positively homogeneous function of degree one in L and $\beta, \beta(x,y)= v_i(x,y)y^i, v_i(x,y)$ is an h-vector in$F^n = (M^n, L)$.

Keywords: Finsler space, (α,β)-metric,h-vector, Berwald connection, Cartan connection, β-change, Rander's change, Projective change.

1. INTRODUCTION

Let $F^n = (M^n, L)$ be a Finsler space,M^n an n-dimensional differentiable manifold and L(x,y) is the metric function. A geodesic on $F^n = (M^n, L)$ which is an extremal of the length integral, is given by the system of differential equation ([8], [9])

$dy^i|dt + 2G^i(x,y) = \tau y^i,$ (1.1)

where$y^i = dx^i|dt, \tau = (d^2s|dt^2)|(ds|dt)$ and $G^i(x,y) = \gamma^i_{jk}(x,y)y^j y^k$are (2)

p-homogeneous function in $y^i, \gamma^i_{jk} = \frac{1}{2}g^{ir}\left(\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk}\right), \partial_j = \partial|\partial x^j$

Let $G^i{}_j = \dot{\partial}_j G^i, G_k{}^i{}_j = \dot{\partial}_k G^i{}_j, \dot{\partial}_k = \partial|\partial y^k$.

The connection coefficients of Berwald connection BΓ are $(G^i_k{}_j, G^i{}_j, 0)$. The h- and v-covariant derivatives of a contravariant vector field X^i with respect to BΓ are given by ([8])

$X^i_{;j} = \partial_j X^i - G^m{}_j\left(\dot{\partial}_m X^i\right) + X^m G_m{}^i{}_j$ (1.2)

$X^i_{.\,j} = \dot{\partial}_j X^i$ (1.3)

BΓ is neither h-metrical nor v-metrical since $g_{ij;k} = -2C_{ijk|0}$ and $g_{ij.k} = 2C_{ijk}$ in terms of the Cartan connection CΓ.

The Ricci identities with respect to BΓare given by ([8])

$X^i_{;j;k} - X^i_{;k;j} = X^h H_h{}^i{}_{jk} - X^i_{.h} R^h_{jk}$ (1.4)

$X^i_{;j.k} - X^i_{.k;j} = X^h G_h{}^i{}_{jk}$ (1.5)

$X^i_{.j.k} - X^i_{.k.j} = 0$ (1.6)

The tensors $H_h{}^i{}_{jk}$ and $G_h{}^i{}_{jk}$ are called the h- and h v- curvature tensors respectively and $R^h{}_{jk}$is the v(h) – torsion tensor of BΓ. In terms of the coefficients $(G_j{}^i{}_{k,} G^i{}_j, 0)$ these tensors are written as. ([8])

$R^h{}_{jk} = Q_{(jk)}(\partial_k G^h{}_j - G_j{}^h{}_m G^m{}_k)$ (1.7)

$H^i_{h\ jk} = Q_{(jk)}\{\partial_k G_h{}^i{}_j - G^m{}_k(\dot{\partial}_m G_h{}^i{}_j) + G_h{}^m{}_j G_m{}^i{}_k\}$ (1.8)

$G_h{}^i{}_{jk} = \dot{\partial}_h G^i_{j\ k}$.

Throughout the paper, we shall use the notations

$$Q_{(ij)}(X_{im} Y^m_{jk}) = X_{im} Y^m_{jk} - X_{jkm} Y^m_{ik}$$

and $A_{(i,j,k)}(X^i Y_{jk}) = (X^i Y_{jk} + X^j Y_{ki} + X^k Y_{ij})$

Here $G_h{}^i{}_{jk}$ is symmetric in subscripts and$G_h{}^i{}_{j0}$ where '0' denotes the contraction with respect to the supporting element y^k throughout this paper.

Matsumoto [7] has introduced the metric

"L(x, y) = L(x, y) + $\beta\ (x, y)$, (1.9)

$\beta\ (x, y) = v_i(x) y^i$

Hashiguchi and Ichijyo [3] called it a Rander's change.

The change

$'$L $(x, y) = L^2(x, y)\ |\beta\ (x, y)$ (1.10)

is called a Kropina change ([11])

Shibata ([13]) has introduced a β-change by

*L = f(L, β), (1.11)

$\beta = v_i(x) y^i$ and f is a positively homogeneous function of degree one in L and β. If L is a Riemannian metric, then *L = $f(\alpha, \beta)$ becomes (α, β) metric. Many authors ([2], [4], [5], [10], [12], [14]) studied the properties of this metric with different physical and mathematical aspects. In all these works, $v_i(x)$ are assumed to be a function of coordinates only.

During the study of conformal transformation of Finsler space, Izumi ([6]) introduced an h-vector which is defined by $v_i|_j = 0$, L$C_i{}^h{}_j v_h = K h_{ij}, K = \frac{L\, C^i v^i}{(n-1)}, C^h_{ij} = g^{hk} C_{ijk}, C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$is Cartan's C-tensor, $C^i =$

$g^{jk}C^i_{jk}, h_{ij} = L(\partial^2 L|\partial y^i \partial y^j)$is the angular metric tensor, $v_i|_j$ is the v-covariant derivative with respect to the Cartan connection CΓ $(F^i_{jk}, N^i_k, C^i_{jk})$, [9]

$$v_i|_j = \dot{\partial}_j v_i - v_m C_i{}^m{}_j$$

Thus the h-vector $v_i(x, y)$ is not only a function of coordinates but it is also a function of directional argument satisfying $\dot{\partial}_j v_i = (K|L)h_{ij}$.

Singh and Srivastava ([15]) studied the properties of Finsler space with a change.

$\bar{L} = f(L, \beta)$. (1.12)

where $\beta(x, y) = v_i(x, y)y^i, v_i(x, y)$ is an h-vector in F^n. We shall call this change ([1.12]) a generalized β-change by an h-vector.

In the present paper, we shall determine the conditions under which a geodesic of a Finsler space $F^n = (M^n, L)$is also geodesic of the Finsler space $\bar{F}^n = (M^n, \bar{L})$

2. THE FINSLER SPACE $\bar{F}^n = (M^n, \bar{L})$

Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be the Finsler spaces defined on the same manifold M^n, where L is obtained by a change

$\bar{L} = f(L, \beta)$. (2.1)

β(x, y) = $v_j(x, y)y^j$, $v_j(x, y)$ is an h-vector in $F^n = (M^n, L)$ and $f(L, \beta)$ is a positively homogeneous function of degree one in L and β.

The terminology and notations are referred to Matsumoto's book ([9]) unless otherwise stated.

The quantities of Finsler spaces $\bar{F}^n$are denoted by barred symbols.

If l_i, g_{ij},h_{ij} and C_{ijk} denote the normalized element of support, the metric tensor, the angular metric tensor and Cartan's C-tensor of F^n respectively, then these quantities of $\bar{F}^n = (M^n, \bar{L})$ are given by ([15])

$\bar{l}_i = f_1 l_i + f_2 v_i$ (2.2)

$\bar{h}_{ij} = q' h_{ij} + r_0 m_i m_j$ (2.3)

$\bar{g}_{ij} = q' g_{ij} + q_0 v_i v_j + q_{-1}(v_i y_j + v_j y_i) + q'_{-2} v_i v_j$ (2.4)

$\bar{C}_{ijk} = q' C_{ijk} + q'_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + q_{02} m_i m_j m_k |2$ (2.5)

where we put $f_1 = \partial f|\partial L,\ f_2 = \partial f|\partial \beta,\ f_{11} = \partial^2 f|\partial L \partial L, f_{12} = \partial^2 f|\partial L \partial \beta\ etc,$

$\dot{\partial}_i = \partial \big| \partial y^i, \partial_i = \partial \big| \partial x^i$

$$(2.6)\begin{cases} q = ff_1|L, \quad r = ff_2,\ r_0 = ff_{22} \\ f = f_1L + f_2\beta, Lf_{12} + \beta f_{22} = 0, Lf_{11} + \beta f_{12} = 0, \\ q_0 = r_0 + f_2^2, \quad r_{-1} = ff_{12}|L, \quad q_{-1} = r_{-1} + qf_2|f \\ r_{-2} = f(f_{11} - f_1|L)|L^2, \quad q_{-2} = r_{-2} + q^2|f^2 \\ q' = f(f_1 + Kf_2)|L, \quad q'_{-2} = q_{-2} - Kr|L^3, \\ q'_{-1} = q_{-1} + (K|L)q_0 \ and\ q_{02} = \partial q_0|\partial\beta. \end{cases}$$

$m_i = v_i - \beta y_i|L^2$ is a non vanishing vector orthogonal to the supporting element y^i

The reciprocal tensor $\bar{g}^{ij}$ of $\bar{g}_{ij}$ can be written as ([15])

$\bar{g}^{ij} = (1|q')q^{ij} - u'_0 v^i v^j - u'_{-1}(v^i y^j + v^j y^i) - u'_{-2} y^i y^j$ (2.7)

where $v^i = g^{ij}v_j, v^i = g^{ij}v_j,\ v^2 = g^{ij}v_iv_j,\ \epsilon = v^2 - (\beta^2|L^2)$

$u'_0 = f^2 r_0|L^2 v' q', u'_{-1} = (f^2|q'v'L^2)(q_{-1} + Kf_2^2|L)$

$v' = (f^2|L^2)(q' + \epsilon r_0), u'_{-2} = \dfrac{q'_{-2}}{qq'} - (u'_{-1}|q)(\epsilon q_{-1} - Kr\beta|L^3)$

we shall assume that $q' + \epsilon r_0 \neq 0$ and $q + \epsilon r_0 \neq 0$ for all values of K. From the homogeneity, we have

$$\left.\begin{matrix} r_0\beta + r_{-1}L^2 = 0,\ r_{-1}\beta + r_{-2}L^2 = -q, \\ q_0\beta + q_{-1}L^2 = r,\ r\beta + qL^2 = f^2, \\ q_{-1}\beta + q_{-2}L^2 = 0 \end{matrix}\right\} \qquad (2.8)$$

3.RELATION BETWEEN PROJECTIVE CHANGE AND GENERALIZED β-CHANGE BY AN h-VECTOR

For two Finsler spaces $F^n = (M^n, L)$and $\bar{F}^n = (M^n, \bar{L})$, if any geodesic on

$F^n = (M^n, L)$ is also a geodesic on $\bar{F}^n = (M^n, \bar{L})$ and vice versa the change L→ $\bar{L}$= f(L, β) of the metric is called projective. A geodesic is given by the differential equation ([8], [9]).

(dy^i|dt) + $2G^i(x,y)$ =τy^i (3.1)

$\tau = (d^2s|dt^2)|(ds|dt),\ G^i(x,y)$ =$\gamma^i_{gk}(x,y)y^jy^k$

are (2)p homogeneous function in y^i.

Consider the Euler Lagrange Differential equation E_i= 0, where

$E_i = (\partial L|\partial x^i) - (d|dt)(\partial L|\partial \dot{x}^i)$ (3.2)

Now Euler Lagrange Differential Equation Now for $\bar{F}^n = (M^n, \bar{L})$ is given by $\bar{E}_i = 0$,

Now

$$\bar{E}_i = (\partial f|\partial x^i) - (d|dt)(\partial f|\partial \dot{x}^i)$$

$$= f_1\partial_i L + f_2\partial_i\beta - \frac{d}{dt}(f_1\dot{\partial}_i L + f_2\dot{\partial}_i\beta)$$

$$= f_1\left\{\partial_i L - \frac{d}{dt}\left(\dot{\partial}_i L\right)\right\} + f_2\partial_i\beta - \frac{df_1}{dt}l_i - \frac{df_2}{dt}b_i - f_2\frac{db_i}{dt}$$

$$= f_1 E_i - m_i\frac{df_2}{dt} + f_2\left(\partial_i b_j - \partial_j b_i\right)y^j - f_2\dot{\partial}_j b_i\frac{dy^j}{dt}$$

$$= f_1 E_i - m_i\frac{df_2}{dt} + f_2\{b_{j|i} + N^m{}_i\dot{\partial}_m b_j + b_m F_j{}^m{}_i - b_{i|j} - N^m{}_j\dot{\partial}_m b_i - b_m F_i{}^m{}_j\}y^j - f_2(K|L)h_{ij}\frac{dy^j}{dt}.$$

$$= f_1 E_i - m_i\left(\frac{df_2}{dt}\right) + f_2\left(b_{j|i} - b_{i|j}\right)y^j - (K|L)f_2\, h_{ij}\,(dy^j|dt + 2G^j)$$

Using (3.1), we have

$\bar{E}_i = f_1 E_i - m_i(df_2|dt) + 2f_2 F_{oi}$ (3.3)

where $b_{i|j}$ denotes the h −covariant derivative with respect to Cartan connection CΓ

$(F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$

$F_{ji} = \frac{1}{2}(b_{j|i} - b_{i|j}),\ F_{oi} = F_{ji}y^j$

Hence $f\bar{E}_i = ff_1 E_i - m_i f(df_2|dt) + 2ff_2 F_{oi}$ (3.4)

Now $\frac{df_2}{dt} = \frac{\partial f_2}{\partial L}.\frac{dL}{dt} + \frac{\partial f_2}{\partial\beta}.\frac{d\beta}{dt}$

$$= f_{21}\left(\partial_i L\,\frac{dx^i}{dt} + \dot{\partial}_i L\,\frac{dy^i}{dt}\right) + f_{22}\left(\partial_i\beta\frac{dx^j}{dt} + \dot{\partial}_j\beta\frac{dy^j}{dt}\right)$$

or $f\frac{df_2}{dt} = \frac{1}{2}ff_{22}\left(b_{i|j} + b_{j|i} + 2b_m F_i{}^m{}_j\right)y^i y^j$

$$-(\beta|L)ff_{22}\,\partial_i L\,y^i + ff_{22}\frac{dy^j}{dt}(b_j - \frac{\beta}{L}l_j)$$

$= r_0\,E_{00} + r_0\,2b_m G^m - \frac{\beta}{L}r_0(\partial_i L)y^i + r_0\,(b_j - \frac{\beta}{L}l_j)\frac{dy^j}{dt}$ (3.5)

where $E_{00} = E_{ij}y^i y^j,\ E_{ij} = b_{i|j} + b_{j|i}$

using the relation $\frac{dy^r}{dt} = y\partial_s y_r + g_{rs}\frac{dy^s}{dt}$ (3.6)

the above equation reduces to.

$f\frac{df_2}{dt} = r_0 E_{00} + Lr_0 E_r m^r$ (3.7)

Hence from (3.4) and (3.7), we have

$$f\bar{E}_i = ff_1 E_i - Lr_0 E_r m^r m_i - r_0 E_{00} + 2rF_{oi}$$

or $f\bar{E}_i = LqE_i - Lr_0 E_r m^r m_i + B_i,$ (3.8) where $B_i = 2rF_{oi} - r_0E_{00}$ (3.9)

THEOREM (3.1) A generalized β −change by an h-vector is projective iff $B_i = 0$.

Proof : Let the generalized β −change by an h-vector is projective. Then $E_i = 0$ implies $\bar{E}_i = 0$ and hence we have $B_i = 0$ by (3.8).

Conversely if $B_i = 0$ then from (3.8) $E_i = 0$ implies $\bar{E}_i = 0$ Again from (3.8) if$\bar{E}_i = 0$ and $B_i = 0$ then

we have

$L\, qE_i + r_0L\, m_i m^s E_s = 0$ (3.10)

Contracting (3.10) by y^j, we have

$E_s m^s = 0$ since $q + \epsilon r_0 \neq 0$

$$\therefore E_s = 0$$

From the above theorem, we have the following results by Shibata ([13]) and Hashiguchi and Ichijyo ([3]).

COROLLARY (3.1) A β-change is projective iff $2rF_{oi} = r_0E_{00}m_i$

COROLLARY (3.2) A Rander's change is projective iff b_i is gradient of some scalar function.

Definition(3.1) ([8]) If there exists a projective change $L \to \bar{L}$ of a Finsler space $F^n = (M^n, L)$ such that the Finsler space $\bar{F}^n = (M^n, \bar{L})$ is a locally Minkowski space, $F^n$is called projectively flat and this change $L \to \bar{L}$ is an adapted projective change.

THEOREM (3.2) Let the generalized β-change by an h-vector ([1.12]) is projective and L is Minkowskian, then the Weyl torsion tensor $\bar{W}^i{}_{j\,k}$ and the Douglas tensor $\bar{D}_j{}^i{}_{kl}$of $\bar{F}^n$ vanish. Hence F^n with n >2 is projectively flat.

Proof The Weyl torsion tensor is given by ([81]).

$$W^i{}_{jk} = R^i{}_{jk} + \frac{1}{n+1} Q_{(jk)}\{y^i H_{jk} + \delta^i_j H_k\}$$

where $H_{jk} = H_i{}^i{}_{jk}$ and $H_k = \frac{1}{n-1}(nH_{0k} + H_{k0})$ Since $F^n$is Minkowskian then $H_j{}^i{}_{kl} = 0$ and therefore $H_{jk} = H_k = 0$

Hence $W^i{}_{jk} = 0$

Since $W^i{}_{jk}$ is invariant under a projective change hence $\bar{W}^i{}_{j\;k} = 0$.

The Douglas Tensor $D_j{}^i{}_{kl}$ is given by $D_j{}^i{}_{kl} = G_j{}^i{}_{kl} - \{y^i G_{jk.l} + A_{(j,k,l)}(\delta^i_j G_{kl})\}|(n+1)$

Since F^n is Minkowskian, then $G_j{}^i{}_{kl} = 0$ and so $G_{kl} = 0$ Hence $D_j{}^i{}_{kl} = 0$Since $D_j{}^i{}_{kl}$ is invariant under a projective change hence we have $\bar{D}_j{}^i{}_{kl} = 0$.

Since $W^i{}_{jk} = 0$, $D_j{}^i{}_{kl} = 0$ and n >2, hence F^n is projectively flat. ([8])

THEOREM (3.3) If we suppose that generalized β-change by an h-vecter is projective and L is Riemannian, then Douglas tensor $\bar{D}_j{}^i{}_{kl}$ of $\bar{F}^n$ vanishes.

Proof: Since F^n is Minkowskian, then $G_j{}^i{}_{kl} = 0$and G_{jk} = 0 Hence $D_j{}^i{}_{kl} = 0$ Since $D_j{}^i{}_{kl}$ is invariant under a projective change, hence $\overline{D}_j{}^i{}_{kl}$ = 0.

THEOREM (3.4) If B_i = 0 then $\bar{F}^n$ is of scalar curvature iff $F^n$is of scalar curvature.

Proof : By Szabo ([17]), a Finsler space is of scalar curvature iff the Weyl torsion tensor $W^i{}_{jk}$vanishes identicallyLet B_i=0, then due to Theorem (3.1) generalized β-change by an h-vector is projective. Let F^n be of scalar curvature, then $W^i{}_{jk}$k = 0 But $\overline{W}^i{}_{jk} = W^i{}_{jk}$ = 0 Hence $\bar{F}^n$ is of scalar curvature.

In the Riemannian space, scalar curvature means constant curvature.

Thus we have the following Yasuda and Shimada result ([16])

COROLLARY (3.3) If B_i = 0 in a generalized β-change by an h-vector and F^n is Riemannian, then $\bar{F}^n$ is of constant curvature iff F^n is of constant curvature.

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