

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Dpen Access, Peer-reviewed, Refereed Journal

# SCHUR CONVEXITIES OF SIMILAR PRODUCT TYPE AND DUAL FORM OF GENERALIZED HERON MEAN 

${ }^{1}$ Lakshmi Janardhana R C<br>${ }^{1}$ Department of Mathematics, Govt. First Grade College, Vijayanagara, Bangalore

Abstract: In this paper, the different types of Schur convexities of generalized Heron mean, similar product type means and their dual forms in two variables are discussed using strong mathematical induction by grouping of terms.

Index Terms - Heron mean, generalized Heron means, Schur concavity and convexity.

## I. Introduction:

For positive numbers $a, b$, let $I=I(a, b)= \begin{cases}\exp \left[\frac{b \ln b-a \ln a}{b-a}-1\right], & a<b \\ a, & a=b\end{cases}$

These are respectively called the Identric, Logarithmic and Heron means. In [5, 22, 23], V. Lokesha et al. studied extensively and obtained some remarkable results on the weighted Heron mean, the weighted Heron dual mean and the weighted product type means and its monotonicities. Shi et al.[15], discussed the Schurconvexity and Schur-geometric-convexity of a further generalization of the Heronian means given by

$$
H_{p, w}(a, b)= \begin{cases}\left(\frac{a^{p}+w(a b)^{\frac{p}{2}}+b^{p}}{w+2}\right)^{\frac{1}{p}} & \text { if } p \neq 0  \tag{1.4}\\ \sqrt{a b} & \text { if } p=0\end{cases}
$$

Recently, Li et al.[3] discussed the Schur-convexity and Schur-harmonic-convexity of the generalized Heronian means with two positive numbers. In [19, 21], Zhang et al. gave the generalizations of Heron mean, similar product type means and their dual forms. For two variables, the above means are as follows:

$$
\begin{equation*}
I(a, b ; k)=\prod_{i=1}^{k}\left(\frac{(k+1-i) a+i b}{k+1}\right)^{\frac{1}{k}}, \quad I^{*}(a, b ; k)=\prod_{i=0}^{k}\left(\frac{(k-i) a+i b}{k}\right)^{\frac{1}{k+1}} \tag{1.5}
\end{equation*}
$$

$$
\begin{equation*}
H(a, b ; k)=\frac{1}{k+1} \sum_{i=0}^{k} a^{\frac{k-i}{k}} b^{\frac{i}{k}}, \quad H^{*}(a, b ; k)=\frac{1}{k} \sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}} b^{\frac{i}{k+1}} \tag{1.6}
\end{equation*}
$$

Where k is a natural number. Authors have proved that $\mathrm{H}(\mathrm{a}, \mathrm{b} ; \mathrm{k})$ and $\mathrm{I} *(\mathrm{a}, \mathrm{b} ; \mathrm{k})$ are monotonic decreasing functions and $\mathrm{H} *(\mathrm{a}, \mathrm{b} ; \mathrm{k})$ and $\mathrm{I}(\mathrm{a}, \mathrm{b} ; \mathrm{k})$ are monotonic increasing functions with k and also established the following limiting values of these means.

$$
\lim _{k \rightarrow+\infty} I(a, b ; k)=\lim _{k \rightarrow+\infty} I^{*}(a, b ; k)=I(a, b) \text { and } \lim _{k \rightarrow+\infty} H(a, b ; k)=\lim _{k \rightarrow+\infty} H^{*}(a, b ; k)=L(a, b) .
$$

The Schur convex function was introduced by I Schur, in 1923 and it has many important applications in analytic inequalities. In 2003 X.M. Zhang proposed the concept of "Schur-Harmonically convex function" which is an extension of "SchurConvexity function". Schur-geometrically convexity for different means is discussed in [13, 20]. The detailed discussion on convexity and Schur convexity can also be found in ([2][12]).

## II. DEFINITION AND LEMMAS:

In this section, we recall the definitions and lemmas which are essential to develop this paper.
Definition 2.1. [6], [17] $\Omega \subseteq \mathrm{R}^{\mathrm{n}}$ is called symmetric set if $\mathrm{x} \in \Omega$ implies $\operatorname{Px} \in \Omega$ for every $\mathrm{n} \times \mathrm{n}$ permutation matrix $P$.
The function $\phi: \Omega \rightarrow R$ is called symmetric if for every permutation matrix $\mathrm{P}, \phi(\mathrm{P} x)=\phi(\mathrm{x})$ for all $\mathrm{x} \in \Omega$. Lemma 2.1. [24] Let $\Omega \subseteq \mathrm{R}^{\mathrm{n}}$ be symmetric with non-empty interior geometrically convex set $\Omega^{0}$ and let $\phi: \Omega \longrightarrow \mathrm{R}_{+}$be continuous on $\Omega$ and differentiable in $\Omega^{0}$. Then $\phi$ is Schur-geometrically convex (Schurgeometrically concave) on $\Omega$ if and only if $\phi$ is symmetric on $\Omega$ and

$$
\begin{equation*}
\left(\ln x_{1}-\ln x_{2}\right)\left(x_{1} \frac{\partial \phi}{\partial x_{1}}-x_{2} \frac{\partial \phi}{\partial x_{2}}\right) \geq 0(\leq 0) \tag{2.1}
\end{equation*}
$$

holds for any $\mathrm{x}=\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{n}}\right) \in \Omega^{0}$.
Lemma 2.2. [24] Let $\Omega \subseteq \mathrm{R}^{\mathrm{n}}$ be symmetric with non-empty interior set $\Omega^{0}$ and let $\phi: \Omega \rightarrow \mathrm{R}_{+}$be continuous on $\Omega$ and differentiable in $\Omega^{0}$. Then $\phi$ is Schur convex (Schur concave) on $\Omega$ if and only if $\phi$ is symmetric on $\Omega$ and

$$
\begin{equation*}
\left(x_{1}-x_{2}\right)\left(\frac{\partial \phi}{\partial x_{1}}-\frac{\partial \phi}{\partial x_{2}}\right) \geq 0(\leq 0) \tag{2.2}
\end{equation*}
$$

holds for any $\mathrm{x}^{2}=\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{n}}\right) \in \Omega^{0}$.
Lemma 2.3. [24] Let $\Omega \subseteq \mathrm{R}^{\mathrm{n}}$ be symmetric with non-empty interior harmonic convex set $\Omega^{0}$ and let $\phi: \Omega \rightarrow \mathrm{R}_{+}$be continuous on $\Omega$ and differentiable in $\Omega^{0}$. Then $\phi$ is Schur-harmonic convex (Schur-harmonic concave) on $\Omega$ if and only if $\phi$ is symmetric on $\Omega$ and

$$
\begin{equation*}
\left(x_{1}-x_{2}\right)\left(x_{1}^{2} \frac{\partial \phi}{\partial x_{1}}-x_{2}^{2} \frac{\partial \phi}{\partial x_{2}}\right) \geq 0(\leq 0) \tag{2.3}
\end{equation*}
$$

## III. MAIN RESULT:

In this section, the various kinds of Schur convexities and concavities of generalized Heron mean, similar product type means and their dual forms in two variables are discussed using strong mathematical induction [14] with grouping of terms.
Theorem 3.1. Let $\mathrm{a}, \mathrm{b}$ be positive real numbers and k be non- negative integer. Then generalized Heron mean similar to product type $\mathrm{I}(\mathrm{a}, \mathrm{b} ; \mathrm{k})$ ) is
(1) Schur-geometrically convex (concave) for all values of $k$ and $a \leq(\geq) b$.
(2) Schur convex (concave) for all values of k and $\mathrm{a} \leq(\geq) \mathrm{b}$.
(3) Schur-harmonic convex (concave) for all values of $k$ and $a \geq(\leq) b$.

Proof: The proof is established by discussing the following three cases.
Case (i). For a $>\mathrm{b}>0$ and k be non-negative integer, we have the generalized Heron mean similar to product type

$$
\begin{gather*}
I(a, b ; k)=\prod_{i=1}^{k}\left(\frac{(k+1-i) a+i b}{k+1}\right)^{\frac{1}{k}} \\
I(a, b ; k)=\left[\left(\frac{k a+b}{k+1}\right)\left(\frac{(k-1) a+2 b}{k+1}\right)\left(\frac{(k-2) a+3 b}{k+1}\right) \ldots \ldots \ldots \ldots\left(\frac{k a+b}{k+1}\right)^{\frac{1}{k}}\right. \\
\text { Let } \quad \theta=(\ln a-\ln b)\left(a \frac{\partial I}{\partial a}-b \frac{\partial I}{\partial b}\right) \tag{3.1}
\end{gather*}
$$

Now, we shall prove that $\theta \geq 0$ for all positive integral values of k , by strong mathematical induction.

$$
\text { For } k=1 \text {, }
$$

$$
I=\frac{a+b}{2}
$$

Taking logarithm on both sides and differentiating partially w.r.t $a$ and multiplying by $a$, then we have

Similarly,

$$
\begin{aligned}
& a \frac{\partial I}{\partial a}=\frac{a}{a+b} \\
& b \frac{\partial I}{\partial b}=\frac{b}{a+b}
\end{aligned}
$$

Then

$$
\theta=I(\ln a-\ln b)\left(\frac{a-b}{a+b}\right) \geq 0
$$

For $\mathrm{k}=2, \quad \theta=2 I(\ln a-\ln b)\left(\frac{a^{2}-b^{2}}{(2 a+b)(a+2 b)}\right) \geq 0$
For $\mathrm{k}=3, \quad \theta=\frac{I(\ln a-\ln b)}{3}\left(\frac{6\left(a^{2}-b^{2}\right)}{(3 a+b)(a+3 b)}+\frac{a-b}{a+b}\right) \geq 0$
For $\mathrm{k}=4, \quad \theta=\frac{I(\ln a-\ln b)}{4}\left(\frac{8\left(a^{2}-b^{2}\right)}{(4 a+b)(a+4 b)}+\frac{12\left(a^{2}-b^{2}\right)}{(3 a+2 b)(2 a+3 b)}\right) \geq 0$
Thus, on grouping first and last term, second and second to last term, and so on in $\theta$, we get

$$
\theta=(\ln a-\ln b)\left(a \frac{\partial I}{\partial a}-b \frac{\partial I}{\partial b}\right) \geq 0
$$

Hence $I(a, b ; k)$ is Schur-geometrically convex for all positive integral values of k .
Case (ii). For $a>b>0$ and $k$ be non-negative integer, we have the generalized Heron mean similar to product type

$$
I(a, b ; k)=\prod_{i=1}^{k}\left(\frac{(k+1-i) a+i b}{k+1}\right)^{\frac{1}{k}}
$$

$$
\begin{align*}
I(a, b ; k)= & {\left[\left(\frac{k a+b}{k+1}\right)\left(\frac{(k-1) a+2 b}{k+1}\right)\left(\frac{(k-2) a+3 b}{k+1}\right) \ldots \ldots \ldots \ldots\left(\frac{k a+b}{k+1}\right)\right]^{\frac{1}{k}} } \\
& \text { Let } \quad \theta=(a-b)\left(\frac{\partial I}{\partial a}-\frac{\partial I}{\partial b}\right) \tag{3.2}
\end{align*}
$$

Now, we shall prove that $\theta \leq 0$ for all positive integral values of $k$, by strong mathematical induction

$$
\text { For } k=1, \quad I=\frac{a+b}{2}
$$

Taking logarithm on both sides and differentiating partially w.r.t $a$ and $b$, then we have

Similarly,

$$
\frac{\partial I}{\partial a}=\frac{I}{a+b}
$$

$$
\frac{\partial I}{\partial b}=\frac{I}{a+b}
$$

Then

$$
\theta=(a-b)\left(\frac{\partial I}{\partial a}-\frac{\partial I}{\partial b}\right)=0
$$

For $\mathrm{k}=2$,

$$
\theta=\left(I \frac{(a-b)(b-a)}{2(a+2 b)(2 a+b)}\right) \leq 0
$$

For $\mathrm{k}=3$,

$$
\theta=\left(4 I \frac{(a-b)(b-a)}{3(3 a+b)(a+3 b)}\right) \leq 0
$$

For $\mathrm{k}=4, \quad \theta=\frac{I}{4}\left(\frac{9(b-a)}{(4 a+b)(a+4 b)}+\frac{(b-a)}{(3 a+2 b)(2 a+3 b)}\right) \leq 0$
Thus, on grouping first and last term, second and second to last term, and so on in $\theta$, we get

$$
\theta=(a-b)\left(\frac{\partial I}{\partial a}-\frac{\partial I}{\partial b}\right) \leq 0
$$

Hence $I(a, b ; k)$ is Schur concave for all positive integral values of $k$.
Case (iii). For $a>b>0$ and $k$ be non-negative integer, we have the generalized Heron mean similar to product type,

$$
I(a, b ; k)=\prod_{i=1}^{k}\left(\frac{(k+1-i) a+i b}{k+1}\right)^{\frac{1}{k}}
$$

$$
\begin{equation*}
I(a, b ; k)=\left[\left(\frac{k a+b}{k+1}\right)\left(\frac{(k-1) a+2 b}{k+1}\right)\left(\frac{(k-2) a+3 b}{k+1}\right) \ldots \ldots \ldots \ldots\left(\frac{k a+b}{k+1}\right)\right]^{\frac{1}{k}} \tag{3.3}
\end{equation*}
$$

Let $\theta=(a-b)\left(a^{2} \frac{\partial I}{\partial a}-b^{2} \frac{\partial I}{\partial b}\right)$
Now, we shall prove that $\theta \geq 0$ for all positive integral values of $k$, by strong mathematical induction.
For $k=1$

$$
I=\frac{a+b}{2}
$$

Taking logarithm on both sides and differentiating partially w.r.t $a$ and multiplying by $a^{2}$, then we have

Similarly,

$$
\begin{aligned}
& a^{2} \frac{\partial I}{\partial a}=\frac{a^{2}}{a+b} \\
& b^{2} \frac{\partial I}{\partial b}=\frac{b^{2}}{a+b}
\end{aligned}
$$

$$
\begin{gathered}
\theta=I(a-b)^{2} \geq 0 \\
\theta=\frac{I(a-b)}{2}\left(\frac{4\left(a^{3}-b^{3}\right)+3 a b(a-b)}{(a+2 b)(2 a+b)}\right) \geq 0
\end{gathered}
$$

For $\mathrm{k}=2$,

For $\mathrm{k}=3$,

$$
\theta=\frac{I(a-b)}{3}\left(\frac{6\left(a^{3}-b^{3}\right)+8 a b(a-b)}{(a+3 b)(3 a+b)}-(a-b)\right) \geq 0
$$

For $\mathrm{k}=4, \quad \theta=\frac{I(a-b)}{4}\left(\frac{8\left(a^{3}-b^{3}\right)+17 a b(a-b)}{(a+4 b)(4 a+b)}+\left(\frac{12\left(a^{3}-b^{3}\right)+13 a b(a-b)}{(3 a+2 b)(2 a+3 b)}\right)\right) \geq 0$
Thus, on grouping first and last term, second and second to last term, and so on in $\theta$, we get

$$
\theta=(a-b)\left(a^{2} \frac{\partial I}{\partial a}-b^{2} \frac{\partial I}{\partial b}\right) \geq 0
$$

Hence $I(a, b ; k)$ is Schur-harmonic convex for all positive integral values of $k$.
With similar arguments follows the proof of the following theorems.
Theorem 3.2. Let $a, b$ be positive real numbers and $k$ be non- negative integer. Then generalized dual form of Heron mean similar to product type $I^{*}(a, b ; k)$ is
(1) Schur-geometrically convex (concave) for all values of $k$ and $a \geq(\leq) b$.
(2) Schur concave (convex) for all values of $k$ and $a \leq() \geq b$.
(3) Schur-harmonic convex (concave) for all values of $k$ and $a \geq(\leq) b$.

Theorem 3.3. Let $a, b$ be positive real numbers and $k$ be non-negative integer. Then generalized Heron mean $H(a, b ; k)$ is
(1) Schur-geometric convex (concave) for all values of $k$ and $a \geq(\leq) \underline{b}$.
(2) Schur concave (convex) for all values of $k$ and $a \leq(\geq) b$.
(3) Schur-harmonic convex (concave) for all values of $k$ and $a \geq(\leq) b$.

Theorem 3.4. Let $a, b$ be positive real numbers and $k$ be non-negative integer. Then generalized dual form of Heron mean $H^{*}(a, b ; k)$ is
(1) Schur-geometrically convex (concave) for all values of $k$ and $a \geq(\leq) b$.
(2) Schur concave (convex) for all values of $k$ and $a \leq(\geq) b$.
(3) Schur-harmonic convex (concave) for all values of $k$ and $a \geq$ ( $\leq$ ) $b$.

Acknowledgement: The authors are thankful to the referees for their valuable suggestions.

## References:

[1] R. Abu-Saris and M. Hajja Geometric means of two positive numbers, Mathematical Inequalities and Applications, vol. 9, no. 3, pp. 391-406, 2006.
[2] P.S.Bullen, Handbook of Means and Their Inequalities, Kluwer Acad. Publ., Dordrecht, 2003.
[3] Li-Li Fu, Bo-Yan Xi and H. M. Srivastava Schur-Convexity of the generalized Heronian means involved two positive numbers, Taiwanese Journal of Mathematics, Vol. 15, No. 6, pp. 2721-2731, December 2011.
[4] V.Lokesha, K.M.Nagaraja, Naveen Kumar. B and Y-D.Wu, Shur convexity of Gnan mean for positive arguments, Notes on Number theory and discrete mathematics, Vol. 17, (2011), Number 4, 37-41, ISSN 1310-5132.
[5] V. Lokesha, Zhi-Hua Zhang and Yu-dong Wu, Two weighted product type means and its Monotonicities, RGMIA Research Report Collection, 8(1), Article 17, 2005, \{http://rgmia.vu.edu.au/v8n1.html\}.
[6] A.M. Marshall and I.Olkin, Inequalites: Theory of Majorization and Its Application, New York : Academies Press, 1979.
[7] K.M.Nagaraja and P.S.K.Reddy, Logarithmic convexity and concavity of some Double sequences, Scientia Magna, Vol. 7(2), (2011), 78-81.
[8] K. M. Nagaraja, V. Lokesha and S. Padmanabhan, A simple proof on strengthening and extension of inequalities, Adv. Stud. Contemp. Math., 17(1), (2008), 97-103.
[9] K. M. Nagaraja and P. Siva Kota Reddy, A Note on power mean and generalized contra-harmonic mean, Scientia Magna, 8(3), (2012), 60-62.
[10] K. M. Nagaraja and P. Siva Kota Reddy, Double inequalities on means via quadrature formula, Notes on Number Theory and Discrete Mathematics, 18(1), (2012), 22-28.
[11] K. M. Nagaraja and P. Siva Kota Reddy, Stolarsky's extended family type mean values, Bull. of Pure \& Appl. Math., 6(2), (2012), 255-264.
[12] Naveenkumar. B. Sandeepkumar, V. Lokesha, and K. M. Nagaraja, Ratio of difference of means, International eJournal of Mathematics and Engineeting, 100(2008), 932-936.
[13] H.N.Shi, Y.M. Jiang and W.D. Jiang, Schur-Geometrically concavity of Gini Mean,Comp. Math.

Appl., 57(2009), 266-274.
[14] Sominskii, I. S., The Method of Mathematical Induction, D. C. Heath and Company, Boston, 1963.
[15] H.-N. Shi, M. Bencze, S.-H. Wu and D.-M. Li Schur convexity of generalized Heronian means involving two parameters,J. Inequal. Appl.,2008 (2008), Article ID 879273, 1-9.
[16] K. B. Stolarsky The power and generalized logarithmic means, American Mathematical Monthly, vol. 87, pp. 545-548, 1980.
[17] B.Y. Wang, Foundations of Majorization Inequalities, Beijing Normal Univ. Press, Beijing, China, 1990(in Chinese).
[18] R. Webster, Convexity, Oxford University Press, Oxford, New York, Tokyo, 1994.
[19] Zh.-G. Xiao and Zh.-H. Zhang, The Inequalities G 6 L 6 I 6 A in n Variables, J. Ineq. Pure \& Appl. Math., 4 (2) (2003), Article 39, \{http://jipam.vu.edu.au/v4n2/110 02.pdf\}.
[20] X.M. Zang, Geometrically Convex Functions, An’Hui University Press, Hefei, 2004(in Chinese).
[21] Zh.-H. Zhang and Y.-D. Wu, The generalized Heron mean and its dual form, Appl. Math. E-Notes, 5 (2005), 16-23, \{http://www.math.nthu.edu.tw/ amen/\}.
[22] Zhen-Gang Xiao, Zhi-Hua Zhang and V. Lokesha, The weighted Heron mean of several positive numbers, RGMIA Research Report Collection, 8 (3), Article 6, (2005), \{http://rgmia.vu.edu.au/v8n3.html\}.
[23] Zhen-Gang Xiao, V. Lokesha and Zhi-Hua Zhang, The weighted Heron dual mean of several positive numbers, RGMIA Research Report Collection, 8 (4), Article 19, (2005), \{http://rgmia.vu.edu.au/v8n4.html\}.
[24] Zhen-Hang Yang, Schur Harmonic convexity of Gini means, International Mathematical Forum,


