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SCHUR CONVEXITIES OF SIMILAR PRODUCT TYPE AND DUAL FORM OF GENERALIZED HERON MEAN

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Abstract: In this paper, the different types of Schur convexities of generalized Heron mean, similar product type means and their dual forms in two variables are discussed using strong mathematical induction by grouping of terms.

Index Terms - Heron mean, generalized Heron means, Schur concavity and convexity.

I. INTRODUCTION:

For positive numbers
$$a, b$$
, let $I = I(a,b) = \begin{cases} exp \left[\frac{b \ln b - a \ln a}{b - a} - 1 \right], & a < b \\ a, & a = b \end{cases}$ (1.1)
$$L = L(a,b) = \begin{cases} \frac{a - b}{\ln a - \ln b}, & a \neq b \\ a, & a = b \end{cases}$$
 (1.2)
$$H = H(a,b) = \frac{a + \sqrt{ab} + b}{3}$$
 (1.3)

These are respectively called the Identric, Logarithmic and Heron means. In [5, 22, 23], V. Lokesha et al. studied extensively and obtained some remarkable results on the weighted Heron mean, the weighted Heron dual mean and the weighted product type means and its monotonicities. Shi et al.[15], discussed the Schurconvexity and Schur-geometric-convexity of a further generalization of the Heronian means given by

$$H_{p,w}(a,b) = \begin{cases} \left(\frac{a^{p} + w(ab)^{\frac{p}{2}} + b^{p}}{w+2}\right)^{\frac{1}{p}} & \text{if } p \neq 0\\ \sqrt{ab} & \text{if } p = 0 \end{cases}$$
(1.4)

Recently, Li et al.[3] discussed the Schur-convexity and Schur-harmonic-convexity of the generalized Heronian means with two positive numbers. In [19, 21], Zhang et al. gave the generalizations of Heron mean, similar product type means and their dual forms. For two variables, the above means are as follows:

$$I(a,b;k) = \prod_{i=1}^{k} \left(\frac{(k+1-i)a+ib}{k+1} \right)^{\frac{1}{k}}, \qquad I^{*}(a,b;k) = \prod_{i=0}^{k} \left(\frac{(k-i)a+ib}{k} \right)^{\frac{1}{k+1}}$$
(1.5)

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$$H(a,b;k) = \frac{1}{k+1} \sum_{i=0}^{k} a^{\frac{k-i}{k}} b^{\frac{i}{k}}, \qquad H^*(a,b;k) = \frac{1}{k} \sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}} b^{\frac{i}{k+1}}$$
(1.6)

Where k is a natural number. Authors have proved that H(a, b; k) and I * (a, b; k) are monotonic decreasing functions and H* (a, b; k) and I(a, b; k) are monotonic increasing functions with k and also established the following limiting values of these means.

$$\lim_{k \to +\infty} I(a,b;k) = \lim_{k \to +\infty} I^*(a,b;k) = I(a,b) \quad and \quad \lim_{k \to +\infty} H(a,b;k) = \lim_{k \to +\infty} H^*(a,b;k) = L(a,b).$$

The Schur convex function was introduced by I Schur, in 1923 and it has many important applications in analytic inequalities. In 2003 X.M. Zhang proposed the concept of "Schur-Harmonically convex function" which is an extension of "SchurConvexity function". Schur-geometrically convexity for different means is discussed in [13, 20]. The detailed discussion on convexity and Schur convexity can also be found in ([2]-[12]).

II. DEFINITION AND LEMMAS:

In this section, we recall the definitions and lemmas which are essential to develop this paper.

Definition 2.1. [6], [17] $\Omega \subseteq \mathbb{R}^n$ is called symmetric set if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

The function $\phi : \Omega \to R$ is called symmetric if for every permutation matrix P, $\phi(P x) = \phi(x)$ for all $x \in \Omega$. **Lemma 2.1.** [24] Let $\Omega \subseteq \mathbb{R}^n$ be symmetric with non-empty interior geometrically convex set Ω^0 and let $\phi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0 . Then ϕ is Schur-geometrically convex (Schur-geometrically concave) on Ω if and only if ϕ is symmetric on Ω and

$$\left(\ln x_1 - \ln x_2\right) \left(x_1 \frac{\partial \phi}{\partial x_1} - x_2 \frac{\partial \phi}{\partial x_2}\right) \ge 0 (\le 0).$$
(2.1)

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$.

Lemma 2.2. [24] Let $\Omega \subseteq \mathbb{R}^n$ be symmetric with non-empty interior set Ω^0 and let $\phi: \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0 . Then ϕ is Schur convex (Schur concave) on Ω if and only if ϕ is symmetric on Ω and

$$(x_1 - x_2) \left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2} \right) \ge 0 (\le 0).$$
(2.2)

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$.

Lemma 2.3. [24] Let $\Omega \subseteq \mathbb{R}^n$ be symmetric with non-empty interior harmonic convex set Ω^0 and let $\phi: \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0 . Then ϕ is Schur-harmonic convex (Schur-harmonic concave) on Ω if and only if ϕ is symmetric on Ω and

$$(x_1 - x_2) \left(x_1^2 \frac{\partial \phi}{\partial x_1} - x_2^2 \frac{\partial \phi}{\partial x_2} \right) \ge 0 (\le 0).$$
(2.3)

III. MAIN RESULT:

In this section, the various kinds of Schur convexities and concavities of generalized Heron mean, similar product type means and their dual forms in two variables are discussed using strong mathematical induction [14] with grouping of terms.

Theorem 3.1. Let a, b be positive real numbers and k be non-negative integer. Then generalized Heron mean similar to product type I (a, b; k) is

(1) Schur-geometrically convex (concave) for all values of k and $a \le (\ge) b$.

- (2) Schur convex (concave) for all values of k and $a \le (\ge) b$.
- (3) Schur-harmonic convex (concave) for all values of k and $a \ge (\le) b$.

Proof: The proof is established by discussing the following three cases.

Case (i). For a > b > 0 and k be non-negative integer, we have the generalized Heron mean similar to product type

$$I(a,b;k) = \prod_{i=1}^{k} \left(\frac{(k+1-i)a+ib}{k+1} \right)^{\overline{k}}$$

$$I(a,b;k) = \left[\left(\frac{ka+b}{k+1} \right) \left(\frac{(k-1)a+2b}{k+1} \right) \left(\frac{(k-2)a+3b}{k+1} \right) \dots \left(\frac{ka+b}{k+1} \right) \right]^{\overline{k}}$$
Let $\theta = (\ln a - \ln b) \left(a \frac{\partial I}{\partial a} - b \frac{\partial I}{\partial b} \right)$
(3.1)

Now, we shall prove that $\theta \ge 0$ for all positive integral values of k, by strong mathematical induction.

For
$$k = 1$$
, $I = \frac{a+b}{2}$

Taking logarithm on both sides and differentiating partially w.r.t a and multiplying by a, then we have

Similarly,

$$a\frac{\partial I}{\partial a} = \frac{a}{a+b}$$

$$b\frac{\partial I}{\partial b} = \frac{b}{a+b}$$
Then

$$\theta = I\left(\ln a - \ln b\right) \left(\frac{a-b}{a+b}\right) \ge 0$$

Then

For k = 2,
$$\theta = 2I(\ln a - \ln b)\left(\frac{a^2 - b^2}{(2a + b)(a + 2b)}\right) \ge 0$$

 $I(\ln a - \ln b)\left(-6(a^2 - b^2) - a - b^2\right)$

For k = 3,
$$\theta = \frac{I(\ln a - \ln b)}{3} \left(\frac{6(a^2 - b^2)}{(3a + b)(a + 3b)} + \frac{a - b}{a + b} \right) \ge 0$$

For k = 4,
$$\theta = \frac{I(\ln a - \ln b)}{4} \left(\frac{8(a^2 - b^2)}{(4a + b)(a + 4b)} + \frac{12(a^2 - b^2)}{(3a + 2b)(2a + 3b)} \right) \ge 0$$

Thus, on grouping first and last term, second and second to last term, and so on in θ , we get

$$\theta = \left(\ln a - \ln b\right) \left(a \frac{\partial I}{\partial a} - b \frac{\partial I}{\partial b}\right) \ge 0$$

Hence I (a, b; k) is Schur-geometrically convex for all positive integral values of k.

Case (ii). For a > b > 0 and k be non-negative integer, we have the generalized Heron mean similar to

product type
$$I(a,b;k) = \prod_{i=1}^{k} \left(\frac{(k+1-i)a+ib}{k+1}\right)^{\frac{1}{k}}$$
$$I(a,b;k) = \left[\left(\frac{ka+b}{k}\right)\left(\frac{(k-1)a+2b}{k}\right)\left(\frac{(k-2)a+3b}{k}\right)\right]_{k=1}$$

$$I(a,b;k) = \left[\left(\frac{ka+b}{k+1}\right) \left(\frac{(k-1)a+2b}{k+1}\right) \left(\frac{(k-2)a+3b}{k+1}\right) \dots \left(\frac{ka+b}{k+1}\right) \right]^{\frac{1}{k}}$$

Let
$$\theta = (a-b)\left(\frac{\partial I}{\partial a} - \frac{\partial I}{\partial b}\right)$$
 (3.2)

Now, we shall prove that $\theta \leq 0$ for all positive integral values of k, by strong mathematical induction

For
$$k = 1$$
, $I = \frac{a+b}{2}$

Taking logarithm on both sides and differentiating partially w.r.t a and b, then we have

$$\frac{\partial I}{\partial a} = \frac{I}{a+b}$$
$$\frac{\partial I}{\partial I} = \frac{I}{a+b}$$

Similarly,

$$\theta = (a-b) \left(\frac{\partial I}{\partial a} - \frac{\partial I}{\partial b} \right) = 0$$

 $\partial b = a + b$

For k = 2,
$$\theta = \left(I \frac{(a-b)(b-a)}{2(a+2b)(2a+b)}\right) \le 0$$

For k = 3,
$$\theta = \left(4I\frac{(a-b)(b-a)}{3(3a+b)(a+3b)}\right) \le 0$$

For k = 4,
$$\theta = \frac{I}{4} \left(\frac{9(b-a)}{(4a+b)(a+4b)} + \frac{(b-a)}{(3a+2b)(2a+3b)} \right) \le 0$$

Thus, on grouping first and last term, second and second to last term, and so on in θ , we get

$$\theta = \left(a - b\right) \left(\frac{\partial I}{\partial a} - \frac{\partial I}{\partial b}\right) \le 0$$

Hence I(a, b; k) is Schur concave for all positive integral values of k.

Case (iii). For a > b > 0 and k be non-negative integer, we have the generalized Heron mean similar to product type,

$$I(a,b;k) = \prod_{i=1}^{k} \left(\frac{(k+1-i)a+ib}{k+1} \right)^{\frac{1}{k}}$$

$$I(a,b;k) = \left[\left(\frac{ka+b}{k+1} \right) \left(\frac{(k-1)a+2b}{k+1} \right) \left(\frac{(k-2)a+3b}{k+1} \right) \dots \left(\frac{ka+b}{k+1} \right) \right]^{\frac{1}{k}}$$
Let $\theta = (a-b) \left(a^2 \frac{\partial I}{\partial a} - b^2 \frac{\partial I}{\partial b} \right)$
(3.3)

Now, we shall prove that $\theta \ge 0$ for all positive integral values of k, by strong mathematical induction. For k = 1

$$I = \frac{a+b}{2}$$

Taking logarithm on both sides and differentiating partially w.r.t a and multiplying by a^2 , then we have

$$a^{2} \frac{\partial I}{\partial a} = \frac{a^{2}}{a+b}$$
$$b^{2} \frac{\partial I}{\partial b} = \frac{b^{2}}{a+b}$$

Similarly,

For k = 2,
For k = 3,

$$\theta = \frac{I(a-b)}{2} \left(\frac{4(a^3 - b^3) + 3ab(a-b)}{(a+2b)(2a+b)} \right) \ge 0$$

$$\theta = \frac{I(a-b)}{3} \left(\frac{6(a^3 - b^3) + 8ab(a-b)}{(a+3b)(3a+b)} - (a-b) \right) \ge 0$$

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For
$$k = 3$$
,

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For k = 4,
$$\theta = \frac{I(a-b)}{4} \left(\frac{8(a^3-b^3)+17ab(a-b)}{(a+4b)(4a+b)} + \left(\frac{12(a^3-b^3)+13ab(a-b)}{(3a+2b)(2a+3b)} \right) \right) \ge 0$$

Thus, on grouping first and last term, second and second to last term, and so on $\mathrm{in}\,\theta$, we get

$$\theta = (a-b) \left(a^2 \frac{\partial I}{\partial a} - b^2 \frac{\partial I}{\partial b} \right) \ge 0$$

Hence I(a, b; k) is Schur-harmonic convex for all positive integral values of k. With similar arguments follows the proof of the following theorems.

Theorem 3.2. Let *a*, *b* be positive real numbers and *k* be non-negative integer. Then generalized dual form of Heron mean similar to product type $I^*(a, b; k)$ is

- (1) Schur-geometrically convex (concave) for all values of k and $a \ge (\le)b$.
- (2) Schur concave (convex) for all values of *k* and $a \leq () \geq b$.

(3) Schur-harmonic convex (concave) for all values of k and $a \ge (\le)b$.

Theorem 3.3. Let *a*, *b* be positive real numbers and *k* be non-negative integer. Then generalized Heron mean H(a, b; k) is

(1) Schur-geometric convex (concave) for all values of k and $a \ge (\le)\underline{b}$.

(2) Schur concave (convex) for all values of k and $a \le (\ge)b$.

(3) Schur-harmonic convex (concave) for all values of k and $a \ge (\le)b$.

Theorem 3.4. Let *a*, *b* be positive real numbers and *k* be non-negative integer. Then generalized dual form of Heron mean $H^*(a, b; k)$ is

(1) Schur-geometrically convex (concave) for all values of k and $a \ge (\le) b$.

- (2) Schur concave (convex) for all values of k and $a \le (\ge)b$.
- (3) Schur-harmonic convex (concave) for all values of k and $a \ge (\le) b$.

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