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Total Coloring of Comb Related Graphs and Umbrella Graph

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ABSTRACT

An appropriate coloring of vertices and edges is a total coloring of H in which two nearby (or) incident vertices and edges receive unique colors. $\chi_T(H)$ defines the total chromatic number of H, which color combination requires the fewest number of colors in (H). Match schedule, network task efficiency, and art are all using total coloring [4]. For example, one might be attempting to color a graph to create something visually appealing. When working with colored dots, a sculptor, painter (or) light show artist may determine that keeping the same colors separated in a way that corresponds to appropriate coloring is visually pleasant (or) fascinating. As a result, the graph appears to be quite beautiful.

The total coloring of comb graph (P_n^+), middle graph of comb graph $M(P_n^+)$, total graph of comb graph $T(P_n^+)$, square graph of comb graph $(P_n \odot K_1)^2$ and umbrella graph $U_{(m,n)}$ are all investigated in the present study. We also compute the total chromatic number of the above graphs.

Keywords : Total coloring, total chromatic number, comb graph, middle, total, square graph of comb graph and umbrella graph.

1. INTRODUCTION

All the graphs we're investigating are limited, directionless, and simple. Let H = (V(H), E(H)) be a graph with V(H) is the vertex set and E(H) is the edge set. A function C: V(H) → $\{1,2,\dots,k\}$ is an appropriate k-coloring of graph H. For every p,q ϵ E(H), such that c(p) ≠ c(q). The chromatic number $\chi(H)$ represents the minimum of colors. The chromatic number is the lowest estimate k under which the graph G is considered appropriate coloring. There are numerous types of appropriate coloring, including star coloring, list coloring, harmonious coloring, total coloring, and so on. The total coloring of several specific graphs is explored in this research.

A total coloring of H is defined by the function $\phi : U \rightarrow K$, where U = V(H) ∪ E(H) and K is a collection of colors that satisfy the condition, Two neighboring (or) incidental vertices and edges receive unique colors. The lowest number of colors required to color appropriately (H) is $\chi_T(H)$ and that is the total chromatic number of H. since $\chi_T(H) \geq \Delta(H)+1$, where Δ(H) is the greatest degree of H, is obvious. If a graph H can be appropriately colored with Δ(H)+1 colors, it is categorized as type-I, and if it can be appropriately colored with Δ(H)+2 colors, it is classified as type-II. In 1965, Behzad [1] and vizing [3] created the concept of total coloring. They've also suggested the concept that every simple graph H has $\Delta(H)+1 \leq \chi_T(H) \leq \Delta(H)+2$. The Total Coloring Conjecture (TCC) is the name given to this concept. For any graph H with the greatest degree of 3, M.Rosenfeld [8] proved the TCC. A.V. Kostochka [9] was able to achieve the total coloring of a multigraph with the greatest degree of four. G.Jayaraman and D.Muthuramakrishnan [11] determined the overall chromatic number of double star graph groups. O.V Borodin [16] demonstrated the TCC for planar graphs with the greatest degree of ≥ 9.

The primary goal of this work is to determine the total coloring of the comb graph, the middle graph of the comb graph, the total graph of the comb graph, the square graph of the comb graph, and the umbrella graph. An above graph is also divided into Type-I and Type-II.

2. PREMILINARIES

Definition 2.1 : The comb graph is represented by $P_n \odot K_1$. The P_n is a path graph with (n-1) edges and n vertices. The graph is constructed by connecting each vertex in the path with a pendant edge. There are 2n vertices and 2n-1 edges in the comb graph. It's mentioned by (comb n).

Definition 2.2 [12] : The M(H) denotes the middle graph of H. M(H) has a vertex set of V(H) E(H). In M(H), two vertices u_1, u_2 in the vertex set are nearby if one of the following conditions is true :

- In H, u_1 and u_2 are adjacent, and $u_1, u_2 \in$ E(H).
- In H, u_1 and u_2 are incident, and $u_1 \in$ V(H), $u_2 \in$ E(H).

Definition 2.3 [12] : The T(H) represented by the total graph of H. T(H) has the vertex set V(H) ∪ E(H). If one of the following conditions applies, two vertices in the vertex set of T(H) are neighboring in T(H):

- $u_1, u_2 \in$ V(H) and In H, u_1 is adjacent to u_2
- $u_1, u_2 \in$ E(H) and u_1, u_2 in H are adjacent.
- In H, $u_1 \in$ V(H), $u_2 \in$ E(H), u_1 and u_2 are incident.

Definition 2.4 : The graph $H^2 = (V, E^2)$ is the square of the graph H = (V,E). A comb graph's square is a square comb graph. which is denotes the $P_n \odot K_1^2$(or) $(Comb\ n)^2$. The vertex set and edge set of a square graph of comb graph are given by,

V = $\{v_i \cup u_i$: 1≤ i ≤ n}

E =$\{v_i u_i$: 1≤ i≤n}∪$\{v_i v_{i+2}$:1≤ i≤n-2}∪$\{v_i v_{i+1} \cup u_i v_{i+1} \cup v_i u_{i+1}$:1≤ i≤ n-1}

The $P_n \odot K_1^2$ contains 2n vertices and 5(n-1) edges.

Definition 2.5 : An umbrella $U_{(m,n)}$ is the graph formed by combining the centre vertex of a fan graph F_n to a path graph P_n. There are m+n vertices and 2m+n-2 edges in the $U_{(m,n)}$ graph.

Theorem 2.6 [10] : Let $T_n$be the twig graph. Then the total chromatic number of twig graph is $\Delta(T_n)$ + 1.

Theorem 2.7 [10] : The splitting graph of a comb graph is S'(P_n^+). Then the total chromatic number is Δ(S'(P_n^+)) + 1.

Theorem 2.8 [14] : Let F_k be the flower graph. Then it's total chromatic number is, $\chi_T(F_k)$ = 2n + 1, k ≥ 4.

Theorem 2.9 [17] : Let $G_n$be the gear graph. Then it's total chromatic number is, $\chi_T(G_n)$ = 2n+1, for n ≥3.

Theorem 2.10 [17] : Let S_n be the sunlet graph. Then it's total chromatic number is n+2,

$\chi_T(S_n)$ = n+2, for ≥ 3.

3. Results and Discussion

Total coloring of comb related graphs:

Theorem 3.1. Let the comb graph be P_n^+, then the total chromatic number is

Δ(P_n^+) +1, for n ≥ 3.

Proof. Now consider the comb graph's vertex and edge sets as follows:

$$V(P_n^+) = \{ v_i, u_i : 1 \leq i \leq n\} \text{ and}$$

$$E(P_n^+) = \{ v_i u_i : 1 \leq i \leq n\} \cup \{ v_i v_{i+1} : 1 \leq i \leq n-1\}$$

The total coloring of the comb graph is formed. Define a total coloring ϕ :U → K, where U = $V(P_n^+) \cup E(P_n^+)$ and K = {1,2,3,4} is the collection of colors . Let A be the collection of odd numbers and B be the collection of even numbers. The following is the total coloring assignment for these vertices and edges:

$$\phi\,(u_i) = 4,\ \ for\ \ 1 \le\ i\ \le\ n$$

$$\phi\,(v_i) = \begin{cases} 1, & if\ i\ \in A \\ 2, & if\ i\ \in\ B \end{cases}\ \ for\ \ 1 \le\ i\ \le\ n$$

$$\phi\,(v_i u_i\,) = \begin{cases} 2, & if\ \ i\ \in A \\ 1, & if\ i\ \ \in\ B \end{cases}\ \ for 1 \le i \le n$$

$$\phi\,(v_i v_{i+1}) = \begin{cases} 3, & if\ i\ \in A \\ 4, & if\ i\ \ \in\ B \end{cases}\ \ for\ \ 1 \le i \le\ n-1$$

The comb graph P_n^+, is obviously total colored$\Delta\,(\,P_n^+)$ +1 with colors. As a result, $\chi_T(P_n^+) = \Delta\,(\,P_n^+) +$ 1, for n ≥ 3 is calculated, is the total chromatic number of the comb graph.

Illustration 3.1.1: Total coloring of comb graph P_6^+as shown in Fig 1.

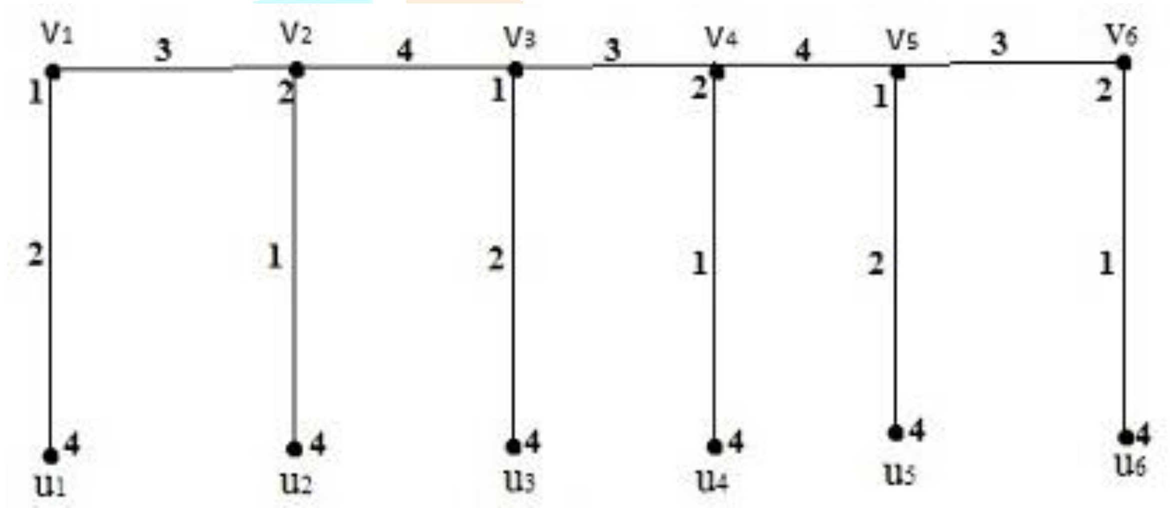

Fig 1 . Total coloring of comb graph P_6^+

Theorem 3.2. Let the middle graph of comb graph be $M(P_n^+)$, then for any n ≥3. Then the total chromatic number is $\Delta(M(P_n^+)) + 1$.

Proof. Now we construct the $V(M(P_n^+))$ and $E(M(P_n^+))$ as follows:

$V(M(P_n^+)) = \{\, v_i, u_i\,, u'_i\!: \le i \le n\} \cup \{v_i'\!: 1 \le i \le n-1\}$

$E\big(M(P_n^+)\big)$= $\{\, v_i u_i{}', u_i{}' u_i\!:\!1 \le i \le n\} \cup \{\, v_i v_i', v_i{}' v_{i+1}, v_i{}' v_{i+1}{}', v_i' u_i', v_i' u_{i+1}'\!: 1 \le i \le$n-1}

$\Delta(M(P_n^+)) = 6$

Define a total coloring $\phi : U \ \rightarrow K$ is then constructed, with U= $V(M(P_n^+)) \cup (E(M(P_n^+))$ and K = {1,2,3,4,5,6,7} is the collection of colors. Let A be the collection of odd numbers and B be the collection of even numbers.

The following is the total coloring assignment for these vertices and edges:

$For\ \ 1 \leq i \leq n$

$$\phi(v_i) = \begin{cases} 4, & if\ i\ \in A \\ 3. & if\ i\ \in\ B \end{cases}$$

$$\phi(u_i') = 6,\ \ \phi\ (u_i) = 2\ ,\ \phi(u_i'u_i) = 1,\ \phi(v_i u_i') = 5\ ,$$

For $1 \leq i \leq n - 1$

$$\phi(v_i' v'_{i+1}) =)\ = \begin{cases} 5, & if\ i\ \in A \\ 6. & if\ i\ \in\ B \end{cases}$$

$$\phi(v_i v_i^{'}) =\ 1,\ \ \phi(v_i^{'} v_{i+1}) = 2\ ,\ \ \phi(v_i{}') = 7\ ,$$

$$\phi(v_i^{'} v_{i+1}) = 2,\ \ \phi(v_i^{'} u_i') = 3,\ \ \phi(v_i^{'} u'_{i+1}) = 4\ ,$$

As can be seen, the graph $M(P_n^+)$ is suitably completely colored using the colors $\Delta(M(P_n^+)) + 1$. Finally, the total chromatic number of middle graph of the comb graph is,

$$\chi_T(M(P_n^+)) =\ \Delta(M(P_n^+)) + 1, \text{ for } n \geq 3.$$

Illustration 3.2.1 : Total coloring of middle graph of comb graph $M(P_3^+)$as shown in Fig 2.

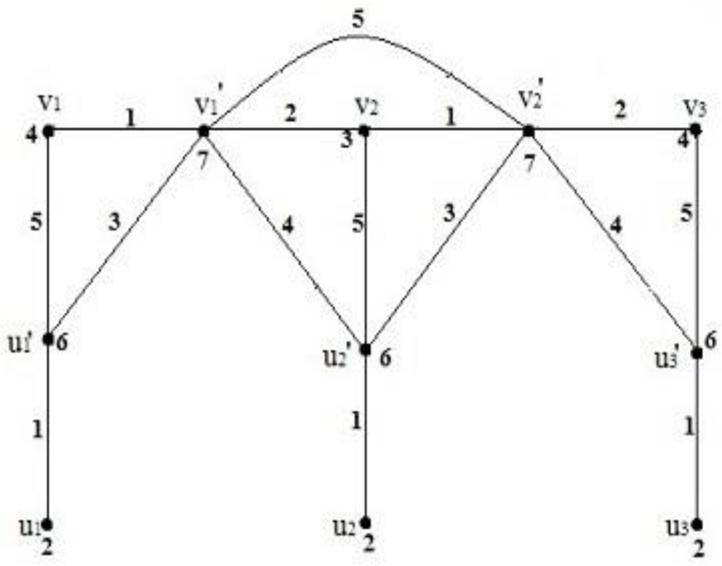

Fig 2. Total coloring of middle graph of comb graph $M(P_3^+)$

Theorem 3.3. The square graph of the comb graph is $P_n \odot K_1^2$. For n ≥ 3, the total chromatic number is $\Delta(P_n \odot K_1^2) + 1$.

Proof. Now we construct the $V(P_n \odot K_1^2)$ and $E(P_n \odot K_1^2)$ as follows:

$V(P_n \circ k_1)^2 = \{ v_i, u_i, u'_i, v'_i : 1 \le i \le n\}$ and

$E(P_n \circ k_1)^2 = \{ v_i u_i, v'_i u'_i : 1 \le i \le n\} \cup \{ v_i v_{i+1}, v_i'^{v'}{}_{i+1} : 1 \le i \le n-2\} \cup$

$\{ v_i v'_i, v_i' v_{i+1}, v_i u'_i, u_i v_i', u'_i v_{i+1}, v'_i u_{i+1} : 1 \le i \le n-1\}$

$\Delta(P_n \odot K_1^2) = 7$

Define a total coloring ϕ : U →K, is then constructed, with U = $V(P_n \odot K_1^2) \cup (P_n \odot K_1^2)$ and K = {1,2,3,4,5,6,7,8} is the collection of colors. Let A be the collection of odd numbers and B be the collection of even numbers.

The following is the total coloring assignment for these vertices and edges:

$For \ 1 \le i \le n$

$$\phi(v_i) = 1, \ \phi(v_i') = 2, \ \phi(u_i') = 4,$$

$$\phi(u_i) = 3, \ \phi(v_i u_i) = 2, \ \phi(v'_i u_i') = 1,$$

For $1 \le i \le n-1$

$$\phi(v_i v'_i) = 3, \quad \phi(v'_i v_{i+1}) = 4, \quad \phi(v_i u'_i) = 7,$$

$$\phi(u_i v'_i) = 8, \quad \phi(v'_i u_{i+1}) = 7, \quad \phi(u'_i v_{i+1}) = 8,$$

For $1 \le i \le n-2$

$$\phi(v_i v_{i+1}) = \begin{cases} 5, & if \ i \in A \\ 6. & if \ i \in B \end{cases}$$

$$\phi(v'_i v'_{i+1}) = \begin{cases} 5, & if \ i \in A \\ 6. & if \ i \in B \end{cases}$$

The total coloring of $P_n \odot K_1^2$ is clearly total colored with $\Delta(P_n \odot K_1^2) + 1$ colors.

As a result, for n ≥ 3, the total chromatic number of square graph of the comb graph is,

$$\chi_T(P_n \odot K_1^2) = \Delta(P_n \odot K_1^2) + 1 .$$

Illustration 3.3.1: Total coloring of square graph of comb graph $(P_n \odot K_1^2)$ as shown in Fig 3.

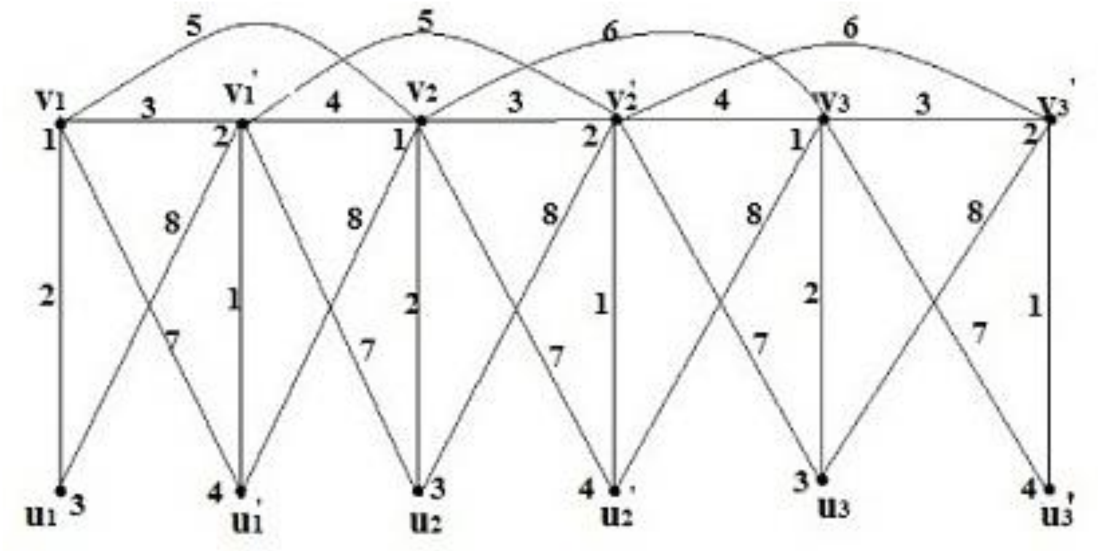

Fig 3. Total coloring of square graph of comb graph $(P_6 \odot k_1)^2$

Theorem 3.4. The total graph of the comb graph is $T(P_n^+)$. For n ≥3 the total chromatic number is, $\chi_T(T(P_n^+)) = \Delta(T(P_n^+)) + 2$.

Proof. We now construct the $V(T(P_n^+))$ and $E(T(P_n^+))$ as follows:

$$V\big(T(P_n^+)\big) = \{ v_i, u_i , u'_i: 1 \le i \le n\} \cup \{v_i': 1 \le i \le n-1\} \text{ and}$$

$$E\big(T(P_n^+)\big) = \{ v_i u_i', u_i' u_i, v_i u_i : 1 \le i \le n\} \cup$$

$$\{ v_i v_i', v_i' v_{i+1}, v_i v_{i+1}, v_i' v'_{i+1}, v_i' u_i', v_i' u_{i+1}' : 1 \le i \le n-1\}$$

$$\Delta(T(P_n^+)) = 6$$

Define a total coloring function ϕ : U → K, is then constructed, with

U = $V(T(P_n^+)) \cup E(T(P_n^+))$ and K = {1,2,3,..., Δ(H)+1} be the collection of colors.

Consider the graph $T(P_3^+)$.

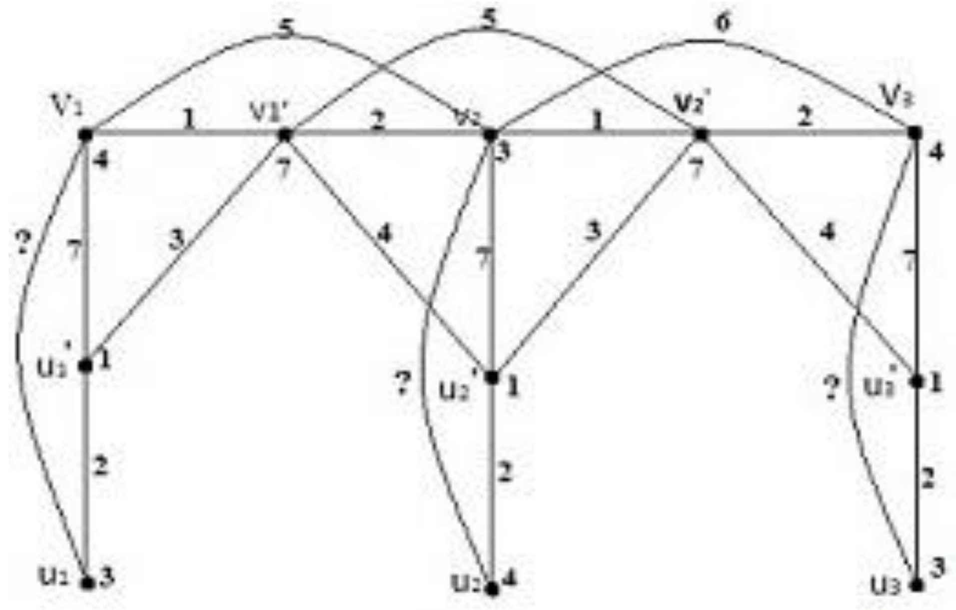

Fig 4. Total coloring of total graph of the comb graph $T(P_n^+)$

The vertices and edges given in fig 4, assigned with available color in set K. 3 more edges namely v_1u_1, v_2u_2, $v_3u_3$need to be colored. Due to proper color condition they need one more color to make the color proper. So $T(P_3^+)$ graph need $\Delta + 2$ colors.

Let A be the collection of odd numbers and B be the collection of even numbers.

The following is the total coloring assignment for these vertices and edges:

$For \;\; 1 \le i \le n,$

$$\phi(u_i') = 1,\; \phi(v_iu_i') = 7\,,\; \phi(u_i'u_i) = 2,\;\; \phi(v_iu_i) = 8\,,$$

$$\phi(v_i) = \begin{cases} 4, & if\;\; i \;\in\; A \\ 3, & if\;\; i \;\in\; B \end{cases}$$

$$\phi\,(u_i) \;\; = \begin{cases} 3, & if\;\; i \;\in\; A \\ 4, & if\;\; i \;\in\; B \end{cases}$$

For 1$\le i \le n-1,$

$\phi(v_i') = 7, \quad \phi(v_iv_i') = 1,\;\; \phi(v_i'\,v_{i+1}) = 2,\;\; \phi(v_i'u_i') = 3\,,\;\; \phi(v_i'u_{i+1}') = 4$

$$\phi(v_iv_{i+1}) = \begin{cases} 5\,, & if\;\; i \in A \\ 6, & if\;\; i \in B \end{cases}$$

$$\phi(v_i'v'_{i+1}) = \begin{cases} 5\,, & if\;\; i \in A \\ 6, & if\;\; i \in B \end{cases}$$

It is the proper total coloring of $T(P_3^+)$ with $\Delta(T(P_n^+))$ colors. As a result, the total chromatic number of total graph of the comb graph is,

$$\chi_T(T(P_n^+)) = \Delta(T(P_n^+)) + 2, \text{ for } n \geq 3.$$

Illustration 3.4.1 : Total coloring of total graph of the comb graph $T(P_3^+)$ as shown in Fig 5.

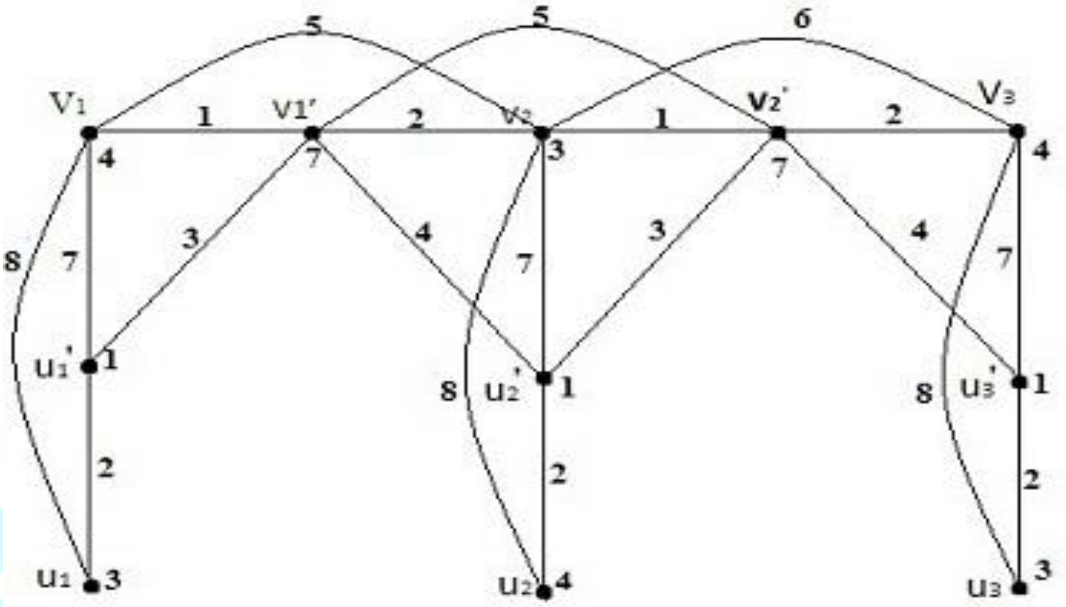

Fig 5. Total coloring of total graph of the comb graph $T(P_3^+)$

Total coloring of umbrella graph:

Theorem 3.5. The umbrella graph is $U_{(m,n)}$. For m ≥ n, the total chromatic number is, $\chi_T(U_{(m,n)})$= m+2.

Proof. The $V(U_{(m,n)})$ and $E(U_{(m,n)})$ are now constructed as regards :

$V(U_{m,n}) = \{u_i, v_j : 1 \leq i \leq m \;\&\; 1 \leq j \leq n\}$ and

$E(U_{m,n}) = \{u_i u_{i+1}, v_j v_{j+1} : 1 \leq i \leq m-1 \;\&1 \leq j \leq n-1\} \cup \{v_1 u_i : 1 \leq i \leq m\}$.

There are m+n vertices and 2m+n-2 edges in the umbrella graph. The total coloring $\phi : U \rightarrow K$, is then constructed, with U =$V(U_{(m,n)}) \cup E(U_{(m,n)})$ and K = {1,2,3,...,m+2} being a collection of colors.

The following is the total coloring assignment for these vertices and edges:

$$\phi(u_i) = \text{i}, \text{ for } 1 \leq i \leq m$$

$$\phi(v_i) = \text{m + i}, \text{ for i = 1}$$

$$\phi(v_j) = \text{j}, \text{ for } 2 \leq j \leq n$$

$$\phi(u_i u_{i+1}) = \text{i + 2}, \text{ for } 1 \leq i \leq m-1,$$

$\phi\ (v_i u_i)$ = m , for i = 1

$\phi\ (v_i u_j) =$ j − i , for i = 1 &2 $\leq j \leq m$

$\phi\ (v_i v_{i+1})$ = m + 2 , for i = 1

$\phi(v_i v_j\ )$ = j + 1 , for $2 \leq i \leq n$ and $3 \leq j \leq n$

The graph $U_{(m,n)}$ is accurately total colored with m+2 colors because the above total coloring rule is evident.

As a result, the umbrella graph $U_{(m,n)}$ of the total chromatic number is,

$$\chi_T(U_{(m,n)}) = m+2, \text{ for } m \geq n.$$

Illustration 3.5.1: Consider the Umbrella graph $(U_{5,3})$ as shown in Fig 6.

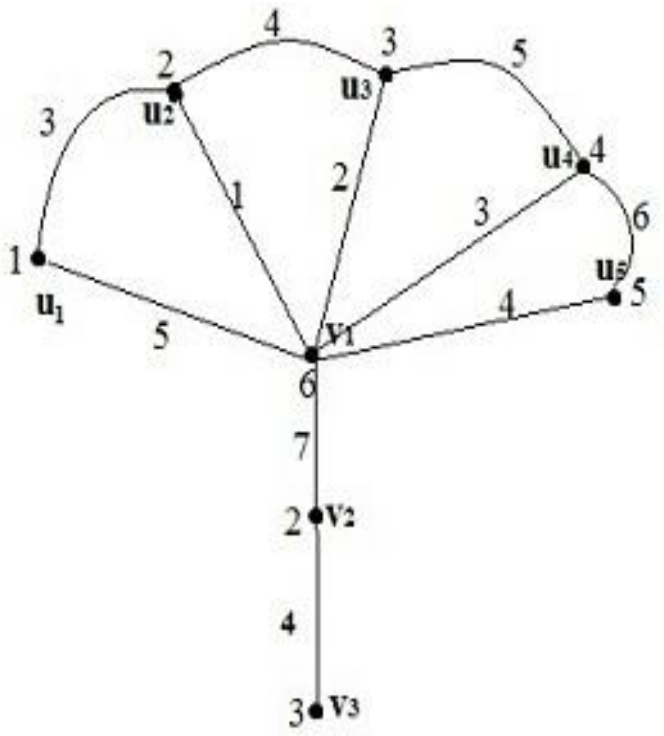

Fig 6. Total coloring of the umbrella graph $(U_{5,3})$

4.CONCLUSION

In this paper, we show that $P_n^+,\ M(P_n^+)\ and\ (P_n \odot K_1^2)$ are Type-I and $T(P_n^+)$ is Type-II, respectively, and the total chromatic number of $U_{(m,n)}$ is m+2.

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