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GRAPH EQUATIONS FOR LINE GRAPHS, MIDDLE GRAPHS, SEMI-TOTAL CLOSED NEIGHBORHOOD BLOCK GRAPHS AND SEMI-TOTAL CLOSED EDGE NEIGHBORHOOD BLOCK GRAPH

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ABSTRACT: Graph equations are those in which the unknowns are graphs. During the past fifty years, the development in these areas has emerged as one of the significant branches in Graph Theory. Let L(G) and M(G) denote the line graph and middle graph respectively. In this paper, we solve the graph equations $L(G) \cong BN_{tc}(H)$, $\overline{L(G)} \cong BN_{tc}(H)$, $M(G) \cong BN_{tc}(H)$, $\overline{M(G)} \cong BN_{tc}(H)$, $L(G) \cong BEN_{tc}(H)$, $\overline{L(G)} \cong BEN_{tc}(H)$, $M(G) \cong BEN_{tc}(H)$, $\overline{M(G)} \cong BEN_{tc}(H)$.

The symbol $\cong$ stands for isomorphism between two graphs.

Keywords: Line graph, middle graph, Semi total closed neighborhood block graphs ,semi total closed edge neighborhood block graphs.

Mathematics subject classification: 05C99

I. Introduction

Kulli and Patil [7] solved the following equations:

$$L(G) \cong e(H), \quad M(G) \cong e(H),$$

$$\overline{L(G)} \cong e(H), \quad \overline{M(G)} \cong e(H)$$

Basavangoud and Mathad [2] solved graph equations:

$$L(G) \cong B_m(H), \quad M(G) \cong B_m(H),$$

$$L(G) \cong B_n(H), \quad M(G) \cong B_n(H),$$

$$J(G) \cong B_m(H), \quad \overline{M(G)} \cong B_m(H),$$

$$J(G) \cong B_n(H), \quad \overline{M(G)} \cong B_n(H).$$

Also Basavanagoud and Mathad (see [3]) obtained solutions for graph equations:

$$L(G) \cong m(H), \quad M(G) \cong m(H),$$

$$L(G) \cong n(H), \quad M(G) \cong n(H),$$

$$J(G) \cong n(H), \quad \overline{M(G)} \cong n(H),$$

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$$J(G) \cong n(H), \quad \overline{M(G)} \cong n(H).$$

Cvetković and Simić [5] solved graph equations $L(G) \cong T(H)$, $\overline{L(G)} \cong T(H)$. Akiyama et.al. [1] solved graph equations $L(G) \cong M(H)$; $M(G) \cong T(H)$; $\overline{M(G)} \cong T(H)$ and $\overline{L(G)} \cong M(H)$.

In this chapter, we solve the following graph equations:

$L(G) \cong BN_{tc}(H)$ $\overline{L(G)} \cong BN_{tc}(H)$

$M(G) \cong BN_{tc}(H)$ $\overline{M(G)} \cong BN_{tc}(H)$

$L(G) \cong BEN_{tc}(H)$ $\overline{L(G)} \cong BEN_{tc}(H)$

$M(G) \cong BEN_{tc}(H)$ $\overline{M(G)} \cong BEN_{tc}(H)$.

That is, we shall find all pairs (G, H) of graphs satisfying equations. Both G and H are called solution of graphs.

For a graph G, let V(G), E(G) and b(G) denote its vertex set, edge set and the set of blocks of G respectively.

Hamada and Yoshimura [6] defined a graph M(G) as an intersection graph $\Omega(F)$ on the vertex set V(G) of any graph G. Let E(G) be the edge set of G and $F = V'(G) \cup E(G)$ where $V'(G)$ indicates the family of one-point subsets of the set V(G). Let $M(G) \cong \Omega(F)$. M(G) is called the middle graph of G.

The open-neighborhood N(u) of a vertex u in V(G) is the set of all vertices adjacent to u.

$N(u) = \{ v \,/\, uv \in E(G) \}$

The closed-neighborhood N[u] of a vertex u in V(G) is given by

$N[u] = \{u\} \cup N(u)$

For each vertex u_i of G, a new vertex u_i' is taken and the resulting set of vertices is denoted by $V_1(G)$.

For a graph G, we defined the semi-total closed neighborhood block graph $BN_{tc}(G)$ of a graph G is defined as the graph having vertex set $V(G) \cup V_1(G) \cup b(G)$ with two vertices are adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex u_i' of $V_1(G)$ and the other to a vertex w_j of G and w_j is in $N[u_i]$ or one corresponds to a vertex u_i of V(G) and the other to a vertex b_i of b(G) and u_i lies on b_i.

The open-neighborhood $N(e_i)$ of an edge e_i in E(G) is the set of edges adjacent to e_i.

$N(e_i) = \{ e_j \,/\, e_i$ and e_j are adjacent in G $\}$.

The closed-neighborhood $N[e_i]$ of an edge e_i in E(G) is given by

$N[e_i] = \{e_i\} \cup N(e_i)$

For each edge e_i of G, a new vertex e_i' is taken and the resulting set of vertices is denoted by $E_1(G)$.

For a graph G, we define the semi-total closed edge neighborhood block graph $BEN_{tc}(G)$ of a graph G as the graph having vertex set $E(G) \cup E_1(G) \cup b(G)$ with two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of $E_1(G)$ and the other to an element e_j of E(G) where e_j is in $N[e_i]$ and the other to a vertex b_i of b(G) and e_i lies on b_i.

In Figure 1, a graph G and its $BN_{tc}(G)$ and $BEN_{tc}(G)$ are shown.

Beineke has shown in [4] that a graph G is a line graph if and only if G has none of the nine specified graphs F_i, $i = 1, 2, \ldots, 9$ as an induced subgraph as stated in the Theorem 6.A and $\overline{F_i}$ is the complement of F_i as in Theorem 6.B. We depict here some of the nine graphs which are useful to extract our later results. These are $F_1 = K_{1,3}$, F_2 and $F_3 = K_5 - x$, where x is any edge of K_5.

Before investigating the solution of the above graph equations, we construct a class of graphs which are useful in our later discussion.

Let $K_{1,n}$, $n \geq 4$ be a star with endedges $\{uu_1, uu_2, \ldots, uu_n\}$. Take $1 \leq m < n$ copies of the star $K_{1,2}$ with endedges $\{v_1w_1, v_1w_2\}$, $\{v_2w_3, v_2w_4\}$ …, $\{v_mw_{2m-1}, v_mw_{2m}\}$. Consider the graph $K_{1,n} \cup mK_{1,2}$ and denote this by $K^0_{n,2}$. Next identify u_1 with w_1 and u_2

with w_2. The resulting graph is denoted by $K^{1}_{n,2}$. Further identifying u_3 with w_3 and u_4 with w_4, we obtain the new graph which is denoted by $K^{2}_{n,2}$ and so on. In this way, we obtain the graph $K^{m}_{n,2}$.

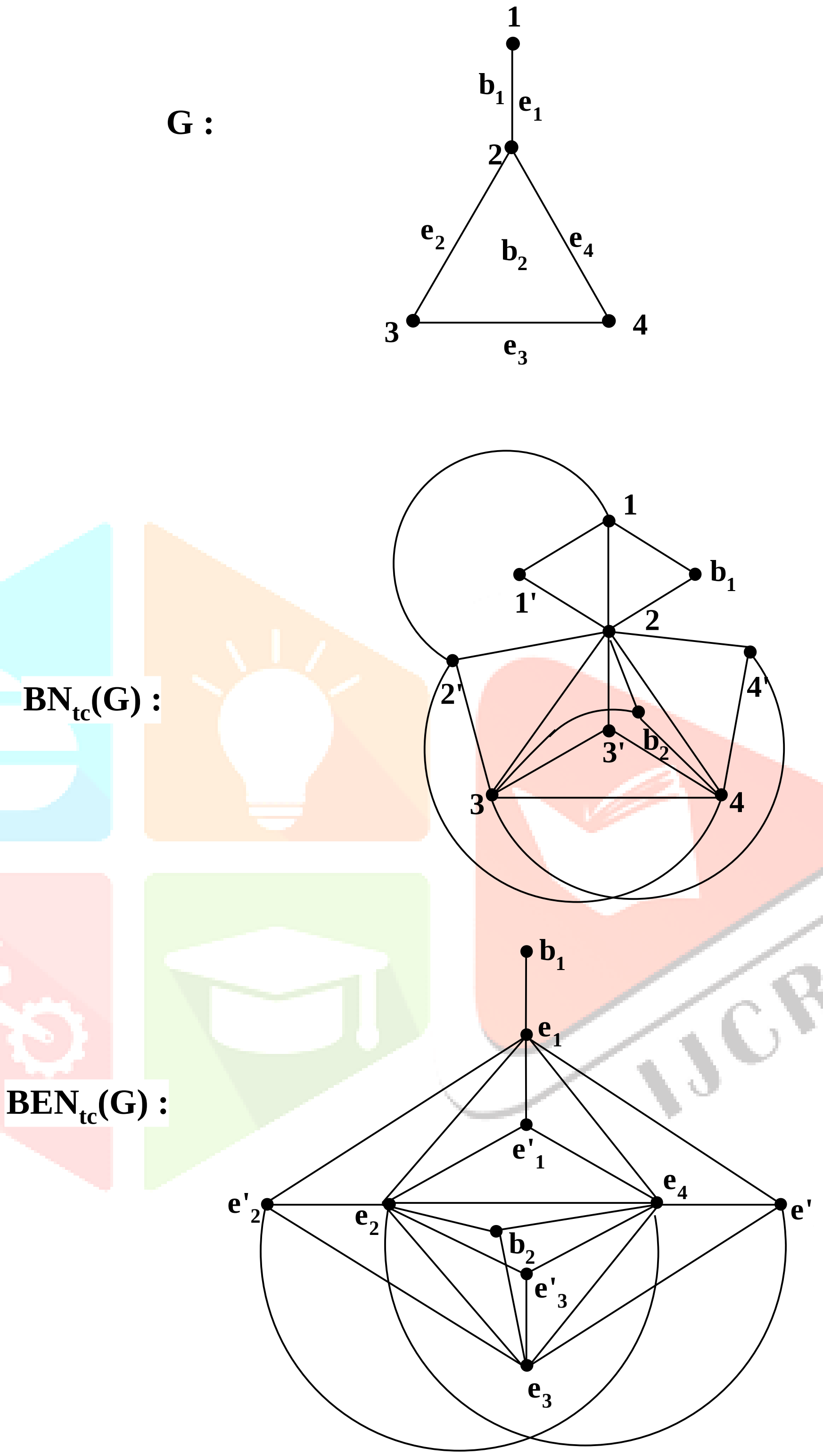


Figure 1.

A graph G^+ is the endedge graph of a graph G if G^+ is obtained from G by adjoining an endedge $u_i u_i'$ at each vertex u_i of G (see[1]). Hamada and Yoshimura have proved in [6] that $M(G) \cong L(G^+)$.

2. THE SOLUTION OF L(G) ≅ BN$_{tc}$(H)

In this case, we observe that if H has an edge then F_1 is an induced subgraph of $BN_{tc}(H)$. Hence H is nK_1, $n \geq 1$. The corresponding G is $nK_{1,2}$.

Thus we obtain the following result.

THEOREM 2.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation $L(G) \cong BN_{tc}(H)$:

$(nK_{1,2}, nK_1)$, $n \geq 1$

3. THE SOLUTION OF $\overline{L(G)} \cong BN_{tc}(H)$

In this case H satisfies the following properties:

i) If H has atleast one edge, then it is connected, since otherwise, $\overline{F_1}$ is an induced subgraph of $BN_{tc}(H)$.

ii) H does not contain cut-vertices, since otherwise, $\overline{F_2}$ is an induced subgraph of $BN_{tc}(H)$.

iii) H does not contain C_n, $n \geq 5$ as an induced subgraph, since otherwise, $\overline{F_2}$ is an induced subgraph of $BN_{tc}(H)$.

iv) H does not contain a vertex which is not adjacent to two mutually adjacent vertices, since otherwise, $\overline{F_2}$ is an induced subgraph of $BN_{tc}(H)$.

v) H is not complete bipartite graph $K_{m,n}$, $m \geq 4$ or $n \geq 4$, since otherwise, F_3 is an induced subgraph of $BN_{tc}(H)$.

We consider the following cases:

Case 1. Assume H is disconnected. Then from observation (i), it follows that H is totally disconnected graph.

For $H = nK_1$, $n \geq 4$, $\overline{F_3}$ is an induced subgraph of $BN_{tc}(H)$. Hence $H = nK_1$, $n \leq 3$.

For n = 1, $H = K_1$ and $G = 2K_2$

For n = 2, $H = 2K_1$ and $G = C_4$

For n = 3, $H = 3K_1$ and $G = K_4$.

Case 2. Assume H is connected. We consider the following subcases:

Subcase 2.1. $H = K_n$. It follows that $(K_{1,n+1} \cup nK_2, K_n)$, $n \geq 2$ and $(K_3 \cup 2K_2, K_2)$ are the solutions.

Subcase 2.2. $H = K_{m,n}$. From observation (v), it follows that H is $K_{2,2}$ or $K_{2,3}$ or $K_{3,3}$.

For $H = K_{2,2}$, G is the graph shown in Fig. 2 (a)

For $H = K_{2,3}$, G is the graph shown in Fig. 2 (b)

For $H = K_{3,3}$, G is the graph shown in Fig. 2 (c)

Subcase 2.3. H is neither a complete graph nor a complete bipartite graph. It follows from observations (ii), (iii), and (iv) that H is a block not containing a vertex which is not adjacent to two mutually adjacent vertices. Therefore, in this case, $H = K_n - \{e_1, e_2, \ldots, e_m\}$, where $1 \leq m < n$. The corresponding G is the graph $K^{m}_{n+1,2} \cup (n-2m)K_2$.

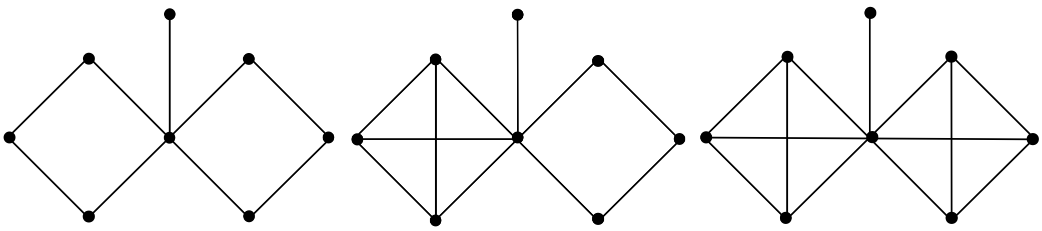
Hence the graph equation 3 is solved and we obtain the following:

THEOREM 3.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation $\overline{L(G)} \cong BN_{tc}(H)$:

$(2K_2, K_1)$; $(C_4, 2K_1)$; $(K_4, 3K_1)$; $(K_{1,n+1} \cup nK_2, K_n)$, $n \geq 2$;

$(K^{m}_{n+1,2} \cup (n-2m)K_2, K_n - \{e_1, e_2, \ldots, e_m\})$, $1 \leq m < n$; $(G_1, K_{2,2})$;

$(G_2, K_{2,3})$ and $(G_3, K_{3,3})$, where G_1, G_2 and G_3 are the graphs shown in Fig. 2(a), 2(b) and 2(c) respectively.



(a) (b) (c)

Figure 2.

4. THE SOLUTION OF $M(G) \cong BN_{tc}(H)$

Theorem 2.1 gives solutions of the graph equation $L(G) \cong BN_{tc}(H)$. But there is no solution which is of the form (G^+, H). Hence, there is no solution of the graph equation $M(G) \cong BN_{tc}(H)$.

Thus, we have the following result.

THEOREM 4.1. There is no solution for the graph equation $M(G) \cong BN_{tc}(H)$.

5. THE SOLUTION OF $\overline{M(G)} \cong BN_{tc}(H)$

Theorem 3.1 gives the solution of the graph equation $\overline{L(G)} \cong BN_{tc}(H)$. Among these solutions, $(2K_2, K_1)$ is of the form (G^+, H). Therefore the solution of $\overline{M(G)} \cong BN_{tc}(H)$ is $(2K_1, K_1)$.

Thus, we have the following result.

THEOREM 5.1 There is only solution $(2K_1, K_1)$ of the graph equation $\overline{M(G)} \cong BN_{tc}(H)$.

6. THE SOLUTION OF $L(G) \cong BEN_{tc}(H)$

In this case, we observe that if H has adjacent edges then F_1 is an induced subgraph of $BEN_{tc}(H)$. Therefore, $\left(nK_2^+, nK_2\right)$, $n \geq 1$ is the solution. Hence we have the following result.

THEOREM 6.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation $L(G) \cong BEN_{tc}(H)$:

$$\left(nK_2^+, nK_2\right),\ n \geq 1.$$

7. THE SOLUTION OF $\overline{L(G)} \cong BEN_{tc}(H)$

Any graph H which is a solution of the above equation satisfies the following properties:

i) If H has atleast one edge, then it is connected, since otherwise, $\overline{F_2}$ is an induced subgraph of $BEN_{tc}(H)$.

ii) H does not contain more than one cut-vertex, since otherwise, $\overline{F_1}$ is an induced subgraph of $BEN_{tc}(H)$.

iii) H does not contain cut-vertex which lies on more than two blocks, since otherwise, $\overline{F_1}$ is an induced subgraph of $BEN_{tc}(H)$.

iv) H does not contain cut-vertex which lies on blocks other than K_2, since otherwise, $\overline{F_1}$ is an induced subgraph of $BEN_{tc}(H)$.

v) H does not contain C_n, $n \geq 5$, as a subgraph, since otherwise, $\overline{F_2}$ is an induced subgraph of $BEN_{tc}(H)$.

vi) H does not contain an edge which is not adjacent to two mutually adjacent edges, since otherwise, $\overline{F_2}$ is an induced subgraph of $BEN_{tc}(H)$.

vii) H is not a complete graph K_n, $n \geq 5$, since otherwise, $\overline{F_2}$ is an induced subgraph of $BEN_{tc}(H)$.

viii) H is not a complete bipartite graph $K_{m,n}$, $m \geq 3$ or $n \geq 3$, since otherwise, $\overline{F_2}$ is an induced subgraph of $BEN_{tc}(H)$.

It follows from observation (ii) that H contains atmost one cut vertex. We consider the following cases.

Case 1. If H has the cut-vertex, then from observations (iii) and (iv), $H = K_{1,2}$. The corresponding G is the graph $K_{1,2}^+$ together with an endedge adjoined to a vertex of maximum degree.

Case 2. If H has no cut-vertices, then we consider the following subcases:

Subcase 2.1. $H = K_n$. In this case, it follows from observation (vii) that $(K_{1.2} \cup K_2, K_2)$, $(K_{1.4} \cup 3K_2, K_3)$ and $(K_{7,2}^3, K_4)$ are the solutions.

Subcase 2.2. $H = K_{m,n}$. Then from observation (viii), it follows that $(K_{5,2}^2, K_{2,2})$ and $(G', K_{1,2})$ where G′ is the graph $K_{1,2}^+$ together with an endedge adjoined to a vertex of maximum degree are the solutions.

Subcase 2.3. H is neither a complete graph nor a complete bipartite graph. It follows from observation (vi) that H is a graph which does not contain an edge which is not adjacent to two mutually adjacent edges. In this case $(K_{6,2}^2, K_4 - x)$, where x is any edge of K_4 and $(K_{5,2}^2, C_4)$ are the solutions.

Thus, we have the following result.

THEOREM 7.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation $\overline{L(G)} \cong BEN_{tc}(H)$:

$(G', K_{1,2})$, where G' is the graph $K^{+}_{1,2}$ together with an endedge adjoined to a vertex of maximum degree; $(K_{1,2} \cup K_2, K_2)$; $(K_{1,4} \cup 3K_2, K_3)$; $(K^{3}_{7,2}, K_4)$; $(K^{2}_{6,2}, K_4 - x)$, where x is any edge of K_4; and $(K^{2}_{5,2}, C_4)$.

8. THE SOLUTION OF $M(G) \cong BEN_{tc}(H)$

Theorem 6.1 gives solution of the equation $L(G) \cong BEN_{tc}(H)$ as (nK^{+}_{2}, nK_2), $n \geq 1$. This is in the form (G^{+}, H). Therefore, the solution of $M(G) \cong BEN_{tc}(H)$ is given by (nK_2, nK_2), $n \geq 1$. Thus we have the following theorem.

THEOREM 8.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation $M(G) \cong BEN_{tc}(H)$:

(nK_2, nK_2), $n \geq 1$.

9. THE SOLUTION OF $\overline{M(G)} \cong BEN_{tc}(H)$

Theorem 7.1 gives the solution of the graph equation $\overline{L(G)} \cong BEN_{tc}(H)$. None of these solutions is of the form (G^{+}, H). Therefore, there is no solution of the equation $\overline{M(G)} \cong BEN_{tc}(H)$.

Thus we obtain the following result.

THEOREM 9.1. There is no solution of the graph equation $\overline{M(G)} \cong BEN_{tc}(H)$.

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