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The Sylow Theorems and their Applications

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***Abstract*: In this article Iam discussing the some applications of Sylow theorem and briefly I will go through the converse of Langrage's theorem on finite groups.**

***Keywords*: Groups, Subgroups, Normal subgroups,Cosets, Cyclic group, Simple group, Sylow subgroup.**

1. Introduction

In mathematics, specifically in the field of the finite group theory, the sylow theorems are collection of theorems named after the Norwegian mathematician Pater Ludwig Sylow (1872) that give detailed information about the number of sub groups of fixed order that given finite group contains, the Sylow theorems forms a fundamental part of finite group theory and have important application of finite simple group

The sylow theorem form a fundamental part of group theory and have an implications on finite simple groups. Further this theorems also asserts of lagrange's theorem

2. Introductory definitions

1) *Definition 1*

Cyclic subgroup; Let G be a group with a€G, then <a> = {a^n, n €Z} is subgroup of G is called the cyclic subgroup generated by a' and a' is called Generator.

2) *Definition 2*

Order of an element and group; Let G be a group and a€G .The Order of an element a€G is the least positive integer n such that a^n=e and is denoted by o(a). If no such n exists for a then order of o(a)=∞

Also the number of elements of G is called the order of group G denoted by o(G)

E.g. G = [z,+] then order of G = ∞

G = [Zn,+] then order of G = n

3) *Definition 3*

Let G be a group, if a€G, we say b€G is conjugate of 'a' denoted a ~ b if there exists some x€G such that xax^{-1} = b

- ˜is an equivalence relation on G

4) *Definition 4*

- Given a group and a subgroup N of G .We say that N is normal subgroup of G if it is closed with respect to conjugation that is for any a€N and x€G,we have that xax^{-1}€N

5) *Definition 5*

P group –if order of G is equal to p^n then G is called p-group

E.g. if o(G)=64 =2^6 is called 2-group

- G be a group and K is P –SSG of G then for all g€G the conjugate gKg^{-1} of K is also a P-SSG
- **Langrange's theorem;** if G is a finite group and H is a sub group of G then o(H divides o(G)
- H=<R90> is a subgroup of D4 such that o(H) = 4 then by Langrange's theorem, o(H)/o(D4)=4/8

If d does not divides order of G then G has no subgroup of order d

Eg. 5 does not divides o (Z14) then Z14 has no subgroup of order 5.

B. Converse of Langranges Theorem

i.e., if d divides o(G) then G may or may not have a subgroup of order d

Proof; we will prove that the converse of Lagrange's theorem need not be true with the help of following example

We know that o(A4) = 12 and 6 / o(A4) . Now we show that A4 has no subgroup of order 6

A4={ e,(1 2)(3 4),(1 3)(2 4), (1 4)(2 3),(1 2 3),(1 2 4) ,(1 3 4),(2 3 4),(4 3 2),(4 3 1),(4 2 1),(3 2 1)} and O(A)=12 and A4 has elements of order 1, 2 and 3. Let H is subgroup of A4 of order 6 then H≈Z6 or S3.if H ≈ Z6 then H has elements of order 6, since H is subgroup of A4 then A4 has elements of order 6.But A4 has no element of order greater than 3 then H is not isomorphic to Z6

If H ≈ S3,then H has 1 element of order 1 , 3 elements of order 2, and 2 elements of order 3

H={ e, (1 2)(3 4), (1 3)(2 4),(1 4)(2 3),a,a^{-1}},where a€A4 and o(a)=3

Now H′=={ e, (1 2)(3 4), (1 3)(2 4),(1 4)(2 3) }subset of H

Now we will show that H′ is group with operation of A4.

	E	(12)(34)	(13)(24)	(14)(23)
E	E	(12)(34)	(13)(24)	(14)(23)
(12)(34)	(12)(34)	e	(14)(23)	(13)(24)
(13)(24)	(13)(24)	(14)(23)	E	(12)(34)
(14)(23)	(14)(23)	(13)(24)	(12)(34)	E

Then H′ is group of order 4, H′ is subgroup Of H.

By Lagrange's theorem o(H′)/o(H) => 4/6 but 4 does not divides 6 then supposition is wrong

Hence A4 has no subgroup of order 6

C. Definition Simple group

A group G is said to be simple group if G as exactly two normal subgroups H={e} and H=G

- Q If G is group of order P then G is always simple Hint; if o(G) =P then G≈Zp and Zp has ţ(p) normal sub groups then G has exactly two normal sub groups then G is simple

- Q if G is cyclic group then G is simple
  Hint; need not, we will prove this with the help of the example

Let G =Z15 is cyclic group and Z15 has exactly ţ(15)= ţ(3×5)=(1+1)(1+1)=4 normal subgroups
Then G =Z15 is not simple
e.g. is G =Z4 ×D6 simple
solution ; we know that Z4×{Ro} is normal subgroup of Z4 ×D6 other than H ={e} and H= Z4 ×D6
then Z4 ×D6 is not simple

- note ; if o(G1) >1 and o(G2)>1 then G1×G2 is not simple
  Hint ; G1×{e2} is normal subgroup of G1×G2 other than H=e and H=G then G1×G2 is not simple

D. Caushy's theorem for finite group

If G is a finite group and P is a prime number such that P divides o(G) then there exist an element 'a' different from 'e' such that a^p=e i.e. o(a) = P

- Remark ; If G is a finite group and P is a prime number such that P divides o(G) then there exist an element 'a' different from 'e' such that a^p=e i.e. o(a) = P
- Then G has element of order P ,
- Then G has a cyclic sub group of order P
  and the number of elements of order P in G is equal to the multilple of phi (P)

Example if O(G) =100 and 5/o(G) then G has elements and then number Of elements of order 5 = phi(5) =4

- If o(G) >1 and G is finite then G has subgroup of order P
- Solution; If o(G) >1 then there exists a prime number P such that P/o(G) then by Caushy's theorem G has elements of order p

Then H = <a> is a subgroup of G order P then G has subgroup of order P.

E. Main point of sylow theorem

Motivation for the sylow theorems comes from attempting to determine the validity of the converse of Lagrange's theorem. To review Lagrange's theorem says that the order of any subgroup H of some group G will divide the order of G. The converse of this could then be that if there exists n€N such that n/O(G) then G has some subgroup H such that o(H) =n . The first sylow therorm serves to determines when this converse actually holds and classifies various sub group of group G based on order.The second and third sylow theorems classifies the relations between the some of the subgroups of G that are equal in size.

1) Sylow's Ist theorem

Let G be a group .If P is a prime number and p^n divides o(G) then (G) has a subgroup of order p^n
e.g. if o(G) =100 then G has subgroup of order 5 and 25
Hint order of G =100 and 5/o(G) then G has subgroup of order 5
5^2/o(G) then G has subgroup of order$5^2 = 25$

2) P –sylow's subgroup or P-SSG

Let G be a finite group and p^n /0(G) but p^{n+1} does not divides o (G) then the subgroup of order $p^n$is called p-sylow subgroup or P-SSG
Example; let o(G)= 60 then 2^2/o(G) but 2^{2+1} does not divides o(G) then the subgroupof order 2^2=4 is called 2-sylow subgroup or 2-SSG

- Q; find order of q-SSG in GLn[Fq]

Hint;if G=GLn[Fq] then we know that o(GLn[Fq])=(q^n-q^{n-1})(q^n-q^{n-2}).........(q^n-1) =q^{n-1}(q-1) q^{n-2}(q^2-1)…1(q^n-1) =$q^{(n-1)+(n-2)+\cdots+2+1}$(q-1)($q^2$-1)… q^n-1)
=$q^{n(n-1)\div 2}$. m

where m= q-1)(q^2-1)… q^n-1) and gcd(q,m)=1

Now $q^{n(n-1)/2}$ divides o(GLn{fq}) but$q^{\frac{n(n-1)}{2}+1}$ does not divides o(GLn {fq})
Then GLn{fq} has q-SSG of $q^{n(n-1)/2}$

- Remark; orderof q-SSG in GLn[fq} is same as order of q-SSG inSLn{fq}

Note: If H is subgroup of G and x€G then its conjugate that is xHx^{-1}={xhx^{-1}/h€H}is subgroup of G
Soln; let G be group and H is subgroup of G and e€H then xhx^{-1}= xex^{-1}€xHx^{-1}
Then e= xex^{-1}€xHx^{-1} then it is non empty subset of G
Now let a=xHx^{-1} then a= ,h1 €H
b€ xHx^{-1} then b= $xh2x^{-1}$,h2€H such that ab^{-1}=($xh1x^{-1}$)($xh2x^{-1}$)= xh $1h2^{-1}x^{-1}$=xh′ x^{-1}€xH x^{-1},
then xH x^{-1} is a subgroup of G

Sylow 2 theorem:

Any two P-SSG of G are conjugate that is H and K are two P-SSG of G then there exists x€G such that K = xH x^{-1}

- Example Show that any two 2-SSG of S3 are conjugate

Solution; G =S3={e,(1 2),(1 3),(2 3),(1 2 3),(1 3 2)}
O(S3)=6=2×3,2/o(G) but 2^2 does not divides o(G) then G =S3 has 2-SSG of order 2
That is subgroup of order 2 of S3 is 2-SSG
2-SSG of S3 are H1 ={e,(1 2),} ,H2={e,(1 3)},H3={e,(2 3)}

Now show that H2 and H3 are conjugate, let x= (1 2) € S3 such that (1 2) H2 $(1\ 2)^{-1}$={(1 2)h2$(1\ 2)^{-1}$/h2€H2} Now show that H2 and H3 are conjugate, let x= (1 2) € S3 such that (1 2) H2 $(1\ 2)^{-1}$={(1 2)h2$(1\ 2)^{-1}$/h2€H2}
= (1 2){e,(1 3)} $(1\ 2)^{-1}$={e,(2 3)}=H3 then H3 = (1 2)H2($(1\ 2)^{-1}$
Then H2 and H3 are conjugate

3) Sylow 3 theorem

If G be a finite group the number of (P-SSG) or p-sylow subgroup in G is equal to 1+PK such that 1+PK / o(G) that is np = 1+PK such that 1+PK/O(G) where K = 0,1,2,…

- Example if O(G) =21 then find number of subgroups of Order 3 in G

Solution ifo(G) =21 =3×7 , then 3/o(G) but 3^2 does not divides O(G) then the subgroup of order 3 is 3-SSG then n3 =1+3K such that 1+3K divides O(G)

If K=0 then n3=1 and 1/ O(G) then n3=1 is possible
If K=1 then n3=4 and 4 does not divides O(G) then n3=4 is not possible
If k=2 then n3 =7 and 7 divides 0(G) then n3 =7 is possible
If k=3 then n3 =10 but 10 does not divides n3=10 is not possible
Similarly K=4,5…are not possible for 3-SSG
Then n3 =1 and n3=7 are possible for 3-SSG

- Note; G has unique p-sylow subgroup or p-SSG iff P-SSG is normal

Hint ; let H is P-SSG of G of order p^n and p-SSG is unique , since H is p-SSG of order p^n then p^n divides order of G but p^{n+1} does not divides order of G . now H is subgroup of G then XHx^{-1} ,x€G is a subgroup of G and O(XHx^{-1}) =o(H)=p^n but p^{n+1} does not divides o(G) . Since G has a unique P-SSG then xHx^{-1} =H for all x€G then H is normal sub group of G

Conversely, Let P-SSG is normal. Let H and k are two P-SSG of (G) then by Sylow's 2 theorem there exist x € G such that K = XHx^{-1} (1)

Since P-SSG is normal then XHx^{-1} = H for all x€G (2)
From 1 and 2 we get
K = H then G has unique P-SSG

Example: if 0(G)= 40=2^3×5 and G is abelian now2^3/o(G) but 2^{3+1} does not divides O(G) then G has 2-SSG of order 8, since G is abelian then 2-SSG of G is normal then 2-SSG is unique then G has unique subgroup of order 8

Note: if o(G) = Pq , P<q and G is abelian then G is cyclic
Hint; O(G) =pq and G is abelian
P/o(G) but p^{1+1} does not divides o(G) then the subgroup of order p is P-SSG. Since G is abelian then P-SSG is normal then P-SSG is unique,
Then G has exactly 1 subgroup of order P. Then number of elements of order P in G = phi(P) =P-1 now q/o(G) but q^{1+1} does not divides o(G) then subgroup of order q is q-SSG since G is abelian then q-SSG is normal then q-SSG is unique, then number of elements of order q = phi(q)=q-1 and G has exactly one element of order 1

Total number of elements of order 1, p and q in G = 1+p-1+q-1= P+q-1 < pq =o(G).

Then G has elements of order other than 1, P and q
G has elements of order pq then G is cyclic then G ≈ Zpq

- Q; if o(G) =39 and G is non abelian then find the number of normal subgroup in G

Solution; o(G)=3×13 and G is non abelian then G has one subgroup of order 1 , 13 subgroups of order 3 , one subgroup of order 13 and one subgroup of order 39

Since H= {e} and H=G is always normal subgroups of G then subgroups of order 1 and 39 are normal subgroups

Normal subgroups of order3; 3/o(G) but 3^{1+1}does not divides o(G) then the subgroup of order 3 is 3-SSG and 3-SSG is not unique Then 3-SSG is not normal .
Subgroups of order 13;
13/o(G) but 13^{1+1} does not divides o(G) then the subgroup of order 13 is 13-SSG
Now 13-SSG of G is unique then it is normal
Then G has unique normal subgroup of order 13
Then total number of normal subgroups in G =1+1+1=3

3. Conclusion

1. Converse of Lagrange's theorem need not be true.
2. If G is finite group and prime no. p divides order of G then Group G has elements of order p.
3. A cyclic group need not be simple.
4. Order of q-sylow subgroups in general linear matrix group over finite field Fq is $q^{n(n-1)/2}$.
5. Group G has unique p-sylow subgroup if p-sylow subgroup is normal.
6. If G is abelian and O (G) = pq, p<q then G is cyclic.
7. If G is non abelian group and O (G) = 39 then G has 3 normal subgroups.

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