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FLAT PARALLEL MINKOWSKI SPACE WITH SPECIAL (α,β) METRIC AND CONFORMAL FLATNESS

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Abstract- The purpose of the present paper is to study the flat parallel Minkowski space with special(α,β) metric and conformally flat finsler space with (α,β) metric.

Keyword- (α,β) metrics, Minkowski space, Flat parallel Minkowski space, Conformal Flatness.

Introduction :

In 1991 M. Matsumoto [3] considered a special class of locally Minkowski space with (α,β) metric, called flat parallel Minkowski space and dealt conformally flatness of generalized Kropina space, through this class of locally Minkowski space. Generalized Kropina does not belong to flat-parallel Minkowski space. Also he dealt flat-parallelness of Rander metric, Kropina metric and their generalized form in his paper [3] .

In this paper we have discussed the flat-parallel Minkowski space and flat-parallelness of Finsler space with special (α,β) metrics $\frac{\beta^2}{(\beta-\alpha)}, \frac{\beta^{m+1}}{(\beta-\alpha)^m} (m \neq 0,-1)$ and $\frac{\beta^2}{(\beta-\alpha)}, \frac{\beta^{m+1}}{(\beta-\alpha)^m} (m \neq 0,-1)$

and $\frac{C_1\alpha^2 + C_2\alpha\beta + C_3\beta^2}{(C_3\alpha + C_4\beta)}$. The last one metric includes simultaneously, Rander metric, Matsumoto metric, Riemannian metric etc. and the corresponding spaces belong to this class.

Further we have discussed the conformally flatness of Finsler space with (α,β) metric $\frac{\beta^{m+1}}{(\beta-\alpha)^m}$ in details with its flat-paralleness.

(1). Locally Minkowski space with (α,β) metric : We consider n-dimensional Finsler space with fundamental function $L(\alpha,\beta)$ where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is Riemannian metric and $\beta = b_i(x)y^i$ is differential one form. Here $L(\alpha,\beta)$ is positively homogenous of degree one in α and β.

Kikuchi [4] has given the condition under which the Randers space with $L = (\alpha + \beta)$ and generalized Kropina space with $L = \alpha^{m+1}\beta^m (m \neq 0,-1)$ be locally Minkowski. Matsumoto [3] defined a special class of locally Minkowski space called flat – parallel Minkowski space and he has found the conditions for Randers and generalized Kropina spaces become flat parallel Minkowski spaces.

Let $\gamma^i_{jk}(x)$ christoffel symbols of the Riemannian metric α and $G^i_{jk}(x)$ be the Berwald connection $B\Gamma$ of (α,β) metric L(α,β). Then the difference

(1.1) $B^i_{jk} = G^i_{jk} - \gamma^i_{jk}$ is given by [6].

(1.2) $L_\alpha B^k_{ji} y^j y_k = \alpha L_\beta (b_{j;i} - B^k_{ji} b_k) y^j$ where

$$y_k = a_{ki}y^i, L_\alpha = \frac{\partial L}{\partial \alpha}, L_\beta = \frac{\partial L}{\partial \beta} \; and \; (;)$$

denotes the covariant differentiation with respect to $\gamma^i_{jk}(x)$.

"A Finsler space is called a Berwald space if G^i_{jk} are function of x^i alone. From (1.1) it is obvious that "A Finsler space with (α,β) metric $L(\alpha,\beta)$ is a Berwald space if and only if B^i_{jk} given by(1.1) are functions of x^i alone i.e. $\dot{\partial}_r B^i_{jk} = 0$ where $\dot{\partial}_r$ stands for $\frac{\partial}{\partial y^r}$.

Further, let R^i_{hjk} be the curvature tensor of associated Riemannian space. Then the h-curvature tensor H^i_{hjk} of $B\Gamma$ is given by [6].

(1.3) $$H^i_{hjk} = R^i_{hjk} + Q_{(jk)}\left\{B^i_{hj;k} - B^r_{ok}\dot{\partial}_r B^i_{hj} + B^r_{hj}B^i_{rk}\right\}$$

where $Q_{(Jk)}\{\ldots\ldots\}$ denote the interchange of j and k and their subtraction.

It is well known([2] theorem 24.5) that a Finsler space is locally Minkowski space if and only if it is a Berwald space with vanishing H^i_{hjk} Consequently we have from (1.2).

"A Finsler with (α,β) metric $L(\alpha,\beta)$ is a locally Minkowski space if and only if B^i_{jk} given by (1.1) are functions of x^i alone and curvature tensor R^i_{hjk} of associated Riemannian space is given by

(1.4). $$R^i_{hjk} = -Q_{(jk)}\left\{B^i_{hj;k} - B^r_{ok}\dot{\partial}_r B^i_{hj} + B^r_{hj}B^i_{rk}\right\}$$

(2). Flat Parallel Minkowski space

"A locally Minkowski space with (α,β) metric is called flat parallel if α is locally flat and β has vanishing covariant derivative of b^i i.e. $R^i_{hjk} = 0$ and $b_{j;i} = 0$ ([3])

We quote the following

Theorem A.[3] : A Rander space is locally Minkowski if and only if it is flat parallel locally Minkowski.

Theorem B.[3] : A locally Minkowski space with following (α,β) metric is always flat parallel Minkowski space

(i) $L = C_1\alpha + C_2\beta + \dfrac{\beta^2}{\alpha}, \qquad C_2 \neq 0$

(ii) $L = C_1\alpha + C_2\beta + \dfrac{\alpha^2}{\beta}, \qquad C_1 \neq 0$

(iii) $L = \dfrac{C_1\alpha^2 + C_2\alpha\beta + C_3\beta^2}{\alpha - \beta}$

Theorem C.[3] : A generalized Kropina space with the metric $L = \alpha^{m+1}\beta^{-m} (m \neq 0, -1)$ is locally Minkowski if and only if the change $(\alpha, \beta) \to (\overline{\alpha}, \overline{\beta})$ where $\overline{\alpha} = e^{-mf}\alpha, \overline{\beta} = e^{-(m+1)f}\beta,$ turns the space into a flat parallel Minkowski space where.

$$f = \log\sqrt{b^2}, \; b^2 = a^{ij}b_i b_j$$

Flat parallel Minkowski space with metric $\dfrac{\beta^2}{(\beta-\alpha)}$ and conformal flatness–

Consider the special (α, β) metric given by

(2.1) $$L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha}$$

Then

$$L_\alpha = \frac{\beta^2}{(\beta-\alpha)^2}, \; L_\beta = \frac{\beta(\beta - 2\alpha)}{(\beta-\alpha)^2}$$

Putting $B^k_{ji} y^j y_k = P_{i00} \quad and \quad (b_{j;i} - B^k_{ji} b_k) y^j = Q_{i0}$ in the equation (1.2) we have

(2.2) $$L_\alpha P_{i00} = \alpha L_\beta Q_{i0}$$

It is clear that for locally Minkowski space $P_{i00} \; and \; Q_{i0}$ are polynomials in y^i of degree two and one respectively. Thus if $P_{i00} = 0 \quad and \quad Q_{i0} = 0$ then we have $B^k_{ji} = 0 \; and \; b_{j;i} = 0$. Therefore(1.4) gives $R^i_{hjk} = 0$ Consequently the Finsler space with (α,β) metric for which $B^k_{ji} = 0, \; b_{j;i} = 0$ is flat parallel.

We shall apply this procedure for the metric (2.1). Using (2.1') in (2.2) we have

$\beta P_{i00} = \alpha(\beta - 2\alpha)Q_{i0}$ which can be written as

$$(2.3) \qquad \alpha\beta Q_{i0} - (2\alpha^2 Q_{i0} + \beta P_{i00}) = 0$$

Thus if the Finsler space with metric (2.1) is locally Minkowski then from (2.3), we observe that

βQ_{i0} and $2\alpha^2 Q_{i0} + P_{i00}$ are rational in y^i where as α is irrational function in y^i. Hence from (2.3) we have

$$Q_{i0} = 0, \quad 2\alpha^2 Q_{i0} + \beta P_{i00} = 0$$

which gives $P_{i00} = 0, \;\; Q_{i0} = 0$. Thus the space is flat parallel. Hence we have the following.

Theorem–1 : A Finsler space with metric $L(\alpha,\beta) = \dfrac{\beta^2}{\beta - \alpha}$ is flat parallel if and only if it is locally Minkowski space.

Next we apply this procedure on the metric

$$L = \frac{C_1\alpha^2 + C_2\alpha\beta + C_3\beta^2}{C_4\alpha + C_5\beta}$$

where C's are constant.

Clearly the metric (2.4) is general form of the metric $L = \dfrac{C_1\alpha^2 + C_2\alpha\beta + C_3\beta^2}{\alpha - \beta}$. By homothetic change of α and β the metric (2.4) is transformed to this metric. From (2.4) we have

$$L_\alpha = \frac{C_1C_4\alpha^2 + 2C_1C_5\alpha\beta + \beta^2(C_2C_5 - C_3C_4)}{(C_4\alpha + C_5\beta)^2}$$

$$L_\alpha = \frac{(C_2C_4 - C_1C_5)\alpha^2 + 2C_3C_4\alpha\beta + C_3C_5\beta^2)}{(C_4\alpha + C_5\beta)^2}$$

using these value of L_α and L_β in(2.2) we have

$$-\left[C_2C_4\alpha^2 + \beta^2(C_2C_5 - C_3C_4)\right]P_{i00} + 2C_3C_4\alpha^2\beta Q_{i0} +$$

$$\alpha[\{(C_2C_4 - C_1C_5)\alpha^2 + C_3C_5\beta^2\}Q_{i0} - 2C_1C_5\beta P_{i00}] = 0$$

If Finsler space with metric (2.4) is locally Minowski, P_{i00} and Q_{i0} are polynomial of degree two and one in y^i respectively. Hence (2.5) can be expressed as $P_i + \alpha Q_i = 0,$ where

(2.6) $$P_i = 2C_3C_4\alpha^2\beta Q_{i0} - [C_1C_4\alpha^2 + \beta^2(C_2C_5 - C_3C_4)]P_{i00}$$

(2.7) $$Q_i = [(C_2C_4 - C_1C_5)\alpha^2 + C_3C_5\beta^2]Qi0 - 2C_1C_5\beta P_{i00}$$

The function P_i and Q_i are rational function of y^i, therefore

$$P_i + \alpha Q_i = 0 \quad gives \quad P_i = 0, \qquad Q_i = 0$$

Now we consider the determinant

$$\Delta = \begin{vmatrix} (C_2C_5 - C_3C_4)\beta^2 + C_1C_4\alpha^2 & -2C_3C_4\alpha^2\beta \\ 2C_1C_5\beta & -(C_2C_4 - C_1C_5)\alpha^2 - C_3C_5\beta^2) \end{vmatrix}$$

Case I : If $\Delta \neq 0,$ we have $P_{i00} = 0,\ Q_{i0} = 0$ and the space becomes flat parallel Minkowski space.

Case II : If $\Delta = 0$ we have

$$-C_1C_4(C_2C_4 - C_1C_5)\alpha^4 + (-C_2^2C_4C_5 + C_4^2C_2C_3)$$
$$+ C_5^2C_1C_2 + 2C_1C_3C_4C_5)\alpha^2\beta^2 - C_3C_5(C_2C_5 - C_3C_4)\beta^4 = 0$$

Since α and β are independent functions, the above equation gives

$$C_1C_4(C_2C_4 - C_1C_5) = 0$$

$$C_3C_5(C_2C_5 - C_3C_4) = 0$$

$$-C_2^2C_4C_5 + C_4^2C_2C_3 + C_5^2C_1C_2 + 2C_1C_3C_4C_5 = 0$$

Solving these equation we have the following

(i) $C_1 = C_2 = C_3 = C_4 = C_5 = 0$

(ii) $C_1 = 0,\ C_2C_5 - C_3C_4 = 0$

(iii) $C_3 = 0, C_2C_4 - C_1C_5 = 0$

Condition (i) gives indeterminate form of L so it is not possible. The solution (ii) gives $L = \frac{C_3}{C_5}\beta$. Therefore (2.2) gives $P_{i00} = 0,\ Q_{i0} = 0.$ The solution (iii) gives $L = \left(\frac{C_2}{C_5}\right)\alpha$ which also gives $P_{i00} = 0,\ Q_{i0} = 0.$ Summarizing above all, we have the following.

Theorem-2: A Finsler space with (α,β) metric $L = \frac{C_1\alpha^2 + C_2\alpha\beta + C_3\beta^2}{C_4\alpha + C_5\beta}$ is flat parallel if and only if it is locally Minkowski.

Flat parallel Minkowski space with (α,β) metric $L(\alpha,\beta) = \frac{\beta^{m+1}}{(\beta-\alpha)^m}(m \neq 0,-1)$

Matsumoto [3] considered generalized Kropina space and showed the condition for them to be locally Minkowski. We shall consider the (α,β) metric.

(3.1) $$L = \frac{\beta^{m+1}}{(\beta-\alpha)^m}(m \neq 0,-1)$$

which gives

(3.2) $$L_\alpha = \frac{mL}{(\beta-\alpha)},\ L_\beta = \frac{\{\beta-(m+1)\alpha\}}{\beta(\beta-\alpha)}L$$

Substituting (3.2) in equation (2.2), we get $m\beta P_{i00} = \alpha[\beta-(m+1)\alpha]Q_{i0}$ which can be written as

(3.3) $$[m\beta P_{i00} + (m+1)\alpha^2 Q_{i0}] - \alpha\beta Q_{i0} = 0$$

If Finsler space with metric (3.1) is locally Minkowski, we have $Q_{i0} = 0,\ m\beta P_{i00} + (m+1)\alpha^2 Q_{i0} = 0$ which gives $P_{i00} = 0,\ Q_{i0} = 0$ and therefore the space is flat parallel. Hence we have the following.

Theorem-3 : A Finsler space with (α,β) metric $L = \frac{\beta^{m+1}}{(\beta-\alpha)^m}\ (m \neq 0,-1)$ is flat parallel if and only if it is locally Minkowski.

(4). Conformally Flat Finsler space with (α,β) metric

$$L(\alpha,\beta)=\frac{\beta^{m+1}}{(\beta-\alpha)^m} \quad (m\neq 0,-1)$$

A Finsler space with fundamental function $L(x,y)$ is called conformally flat, if there exist a local coordinate neibourhood (U,x) of any point of underlying manifold and a differentiable function $\sigma(x)$ and U such that $e^{\sigma}L$ and U is locally Minkowski.

Thus a Finsler space with (α,β) metric

$$L=\frac{\beta^{m+1}}{(\beta-\alpha)^m} \quad (m\neq 0,-1)$$

is conformally flat, if it is locally Minkowskian. Therefore from theorem (3) this space becomes conformally flat if and only if it is flat parallel. i.e. $R^i_{hjk}=0 \text{ and } b_{i;j}=0$

Further for an (α,β) metric, lchilyo-Hashiguchi [1] found a conformally invariant symmetric connection M^i_{jk} defined by.

(4.1) $$M^i_{jk}=\gamma^i_{jk}+\delta^i_j M_k+\delta^i_k M_j - M^i a_{jk}$$

where $M_i=\frac{1}{b^2}\left[b_{i;k}b^k-\frac{b^k{}_{;k}b_i}{n-1}\right], b^2=a^{ij}b_i b_j$ and $M^i=a^{ij}M_j$

Let M^i_{hjk} denote the curvature tensor of the connection M^i_{jk} i.e.

$$M^i_{hjk}=Q_{(jk)}\left(\frac{\partial}{\partial x^j}M^i_{hk}+M^i_{pj}M^p_{hk}\right)$$

Also let $\overset{m}{\nabla}$ denote the covariant differentiation with respect to M^i_{jk} then we have

(4.3) $$\overset{m}{\nabla}_j M_i=\frac{\partial M_i}{\partial x^j}-M^p_{ij}M_p$$

Now suppose that the space is conformally flat therefore for (α,β) metric $L=\frac{\beta^{m+1}}{(\beta-\alpha)^m}$ we have

(4.4) $R^i_{hjk}=0$ and $b_{i;j}=0$ using (4.4) in (4.1) $M_i=0$, and $M^i=0$ $M^i_{jk}=\gamma^i_{jk}$ Thus

$M^i_{hjk}=R^i_{hjk}=0$ Hence we have the following.

Theorem 4 : A Finsler space with (α,β) metric $L=\dfrac{\beta^{m+1}}{(\beta-\alpha)^m}(m\neq 0,-1)$ is conformal to a flat parallel Minkowski space.

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