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Investigation of an open problem (Grothendieck's) on tensor product of locally convex spaces

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ABSTRACT

One of the open problem given by **A. Grothendieck** is "*Is every metrizable (M)-space separable* ?". The objective of this paper is to find some classes of locally convex spaces such that their complete tensor product space is metrizable Montel space. Then, investigate the validation of Grothendieck's problem on this new space. Some theorems required to this work will also be proved.

Keywords: Fréchet space; Projective tensor product; (DF)-spaces; Separable; etc.

INTRODUCTION

In the development of the theory of 'Locally Convex Spaces' known Mathematician, at 50th decades of 19th century, A. Grothendieck, L. Schwartz and N. Bourbaki 'A group of French Mathematician' made their significant role(Radenovic et al; **1998**). On the same interval, A. Grothendieck made several OPEN problems on topological spaces. One of the open problem given by **A. Grothendieck** is

Is every metrizable (M)-space separable ? (Radenovic et al; **1998; Problem no. 10**)

The positive reply were given by J. Dieudonnè, and Gottfried Köthe (G. Köthe; Translated by D.J.H. Garling; Topological Vector Spaces-I; Springer; **1969; p. no. 37**) they proved

Theorem-1 Every (FM)-space are separable.

Theorem-1 The projective tensor product of two metrizable semi-Montel spaces is metrizable and semi-Montel. Therefore the complete tensor product $E\widehat{\otimes}_{\pi}F$ of metrizable semi-Montel spaces E and F is Frèchet semi-Montel space. (PEREZ-GARCIA, C., W. H. SCHIKHOF; Locally C.. **2010, Theorem-10.2.10**)

In **1988**, Jari Taskinen(Helsinki), in his paper 'The projective tensor product of Frèchet-Montel spaces, concluded some consequences……

(i) If E and F are Frèchet-Montel spaces, then $E\widehat{\otimes}_\pi F$ is not necessarily a Frèchet-Montel spaces.
(ii) If E and F are Montel (DF)-space, then its complete injective tensor product space $E\widehat{\otimes}_\varepsilon F$ is not a (DF)-space; Moreover, if F is $l^p - space\ or\ L^p - space\ with$ $1 < p < \infty$, also, then, $E\widehat{\otimes}_\varepsilon F$ is not a (DF)-space.
(iii) For suitable Frèchet-Montel spaces E and F, and , for a Banach space X ,
   (a) $L_b(E,\ E_b') = E_b'\widehat{\otimes}_\varepsilon E_b'$; and
   (b) $L_b(F,\ X) = F_b'\widehat{\otimes}_\varepsilon X$ are not (DF)-spaces.

Concluded that, in general, '*The projective tensor product of Frèchet-Montel spaces is not a Frèchet-Montel space*'.

Theorem-3 if E and F are (DF)-spaces, then $E\widehat{\otimes}_\pi F$ is a (DF)-space' and 'when E and F are Montel (DF)-spaces, then $E\widehat{\otimes}_\pi F$ is a Montel (DF)-space'. (Gottfried Kothe; Topological Vector Spaces-II, Springer-Verlag; **1979; Page no.-186**)

Theorem-4 If E and F are metrizable locally convex spaces then $E\widehat{\otimes}_\pi F$ is a Frèchet space. Let A be a compact subset in $\widehat{\otimes}_\pi F$, then, correspond to the sequences (x_n) in E and (y_n) in F, such that both are converging to the origin, and, every element of A can be written in the form

$$\sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n \quad \textbf{with the condition} \quad \sum_{n=1}^{\infty} |\lambda_n| \le 1$$

<u>Proof</u>. Let (U_n) and (V_n) are neighborhood bases, consisting of a decreasing sequences of absolutely convex sets, in E and F respectively. Then, the absolutely convex envelope of

$$\phi(U_n,\ V_n) = \{x_i \otimes y_i\ :\ x_i \in U_n,\ y_i \in V_n\}$$

Form a neighborhood base for the tensor product space $E\otimes F$, therefore, that is metrizable, and hence $E\widehat{\otimes}_\pi F$ is a Frèchet space.

Taking reference to "A.P. & W. Robertson; Topological Vector Spaces; Second Edition; **P. no.- 133; Lemma-2**" there is a sequence (w_n) in $E\otimes F$, convergent at the origin, then, every element $w \in A$ can be written as

$$\sum_{m=1}^{\infty} \mu_m w_m \quad ; \quad \sum_{m=1}^{\infty} |\mu_m| \le 1$$

Since the sequence (w_n) converges to 0($w_n \to 0$), therefore, there be a sequence $r(m) \to \infty$ With $w_m \in W_{r(m)}$ for each m. Then, each $w_m \in (w_n)$ will be written in the form of finite sum as

$$\sum_{i=1}^{n} v_{mi} x_{mi} \otimes y_{mi} \quad with \quad \sum_{i=1}^{n} |v_{mi}| \le 1;\ x_{mi} \in U_{r(m)}, y_{mi} \in V_{r(m)}$$

And hence,

$$\sum_{m=1}^{\infty}\sum_{i=1}^{n}(\mu_m v_{mi})\, x_{mi}\otimes y_{mi}$$

Now, let us relabeling x_{mi} and y_{mi} to form sequences (x_n) and (y_n) respectively, then, these two sequences converging to 0 and then $\mu_m v_{mi}$ relabeled to form a sequence (λ_n) which satisfying to the condition

$$\sum_{n=1}^{\infty}|\lambda_n| \leq 1$$

Then *w* can be expressed in form which is required.

Thus, we can also wrire this theorem in newform as .. …

" If *E* and *F* are metrizable locally convex spaces, then, every point of the completed tensor product space $E\widehat{\otimes}_\pi F$ can be of the form

$$\sum_{n=1}^{\infty}\lambda_n x_n\otimes y_n \quad \text{with the condition} \quad \sum_{n=1}^{\infty}|\lambda_n| \leq 1$$

Where $x_n \to 0$ *and* $y_n \to 0$. Therefore, we can says that

'If *E* and *F* are metrizable locally convex spaces then $E\widehat{\otimes}_\pi F$ is a Frèchet space'

Or

'If *E* and *F* are Frèchet spaces then $E\widehat{\otimes}_\pi F$ is a Frèchet space'.

Now , taking reference of Gottfried Kothe; Topological Vector Spaces-II, Springer-Verlag; **1979; Page no.-185**, we have, by theorem-6.3

Theorem-5 **If *E* and *F* are metrizable locally convex spaces then every compact set in $E\widehat{\otimes}_\pi F$ is contained in a compact set in the form $\overline{\Gamma(C_1\otimes C_2)}$, where C_1 and C_2 consist of null sequences.**

And in **p. no. 186,** we have

Theorem-6 **If C_n and D_n are fundamental sequences of bounded sets in the spaces *E* and *F* respectively, then $\overline{\Gamma(C_n\otimes D_n)}$ is a fundamental sequence of bounded sets in the complete projective tensor product $E\widehat{\otimes}_\pi F$.**

Thus, concluded

Theorem-7 **If *E* and *F* are (DF)-spaces, then the complete projective tensor product spaces $E\widehat{\otimes}_\pi F$ is a (DF)-space.**

and from theorem-5.16, we find

'The complete projective tensor product of Montel (DF)-spaces is Montel (DF)-space' And by Theorem-6.3 , we have

'If *E* and *F* are metrizable locally convex spaces then $E\widehat{\otimes}_\pi F$ is a Frèchet space'

Now, we arrive at the conclusion

If E and F are Montel metrizable locally convex (DF)-spaces then $E\widehat{\otimes}_{\pi}F$ is a Montel Frèchet space(also called (FM)-space).

Since a Frèchet space is metrizable and complete, thus, we can also construct this problem in another way, equivalently,

The final conclusion

" If E and F are Montel Frèchet locally convex (DF)-spaces then $E\widehat{\otimes}_{\pi}F$ is a Montel Frèchet space."

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