IJCRT.ORG ISSN : 2320-2882
INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

A study on some properties of mortality rate function from modified Weibull distribution

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Abstract

Mortality is a key process in ecology and evolution, and much effort is spent on development and application of statistical and theoretical models involving mortality. Mortality takes place in continuous time, and a fundamental representation of mortality risks is the mortality hazard rate, which is the intensity of deadly events that an individual is exposed to at any point in time. In this article, we provide some properties for the mortality rate function from modified Weibull distribution.

Keywords : mortality rate; scale parameter; shape parameter; aging; age-dependent shape parameter; modified Weibull distribution.

Mathematics Subject Classification (2000) 92D10.62F10.62N05.62P10

1 Introduction

The study of aging has traditionally been independently approached at two levels of biological organization : at the individual and sub-individual level by gerontologists interested in the physiology of human aging and at the population level by demographers primarily interested in patterns of survival and mortality in human populations [2].

Fundamental studies of the aging process have lately attracted the interest of researchers in a variety of disciplines, linking ideas and theories from such diverse fields as biochemistry to mathematics [11]. The way to characterize aging is to plot the increase in mortality rate with chronological age.

The fundamental law of population dynamics is the Gompertz law [3], in which the human mortality rate increases roughly exponentially with increasing age at senescence. The Gompertz model is most commonly employed to compare mortality rate between different populations [6].

It is difficult to find an accurate mathematical model based on old-age mortality pattern [10, 8]. Many mathematical models such as the Weibull, the Heligman-Pollard, the Kannisto, the quadratic and the logistic models yet provide poor fits to empirical mortality pattern at very old ages [8]. The sparse data of observed death rates at very old ages put difficulties in the way of finding an accurate mathematical model [8, 9, 13].

In fact, old age mortality pattern significantly fluctuates with age [10, 8, 9]. Some scientists argued that the oldest-age mortality pattern might level off [8, 11] or decline [13, 4]. The levelling off in senescence of exponential mortality rate increase was well demonstrated by Gavrilov and Gavrilova [?]. Such trends are however too puzzling to reconcile the existing theories of aging [9]. For a better estimation of the maximum human life span, it is thus desirable to find a valid mathematical model which overcomes the uncertainty of old-age mortality pattern.

For ages above 90 years, Thatcher et.al. [8] found that the classical Weibull model is inferior to a model described by Kannisto [5]. For this reason, Weon et.al. [12] modified the classical nature of the stretched exponent as a function of age. With this modification, they made an extended Weibull model adoptable at any shape of the empirical human survival curves [12].

In our previous work, in the absence of age specific mortality data, we gave a mathematical justification to predict the maximum human life span from the modified Weibull model, using the age at menopause parameter. The attempt made in this article gives some properties of mortslity rate function.

2 Modified Weibull Distribution

Over the last few decades, a variety of classical distributions have been widely utilised to model data in a variety of fields, including engineering, actuarial, environmental, and medical sciences, biological studies, demographics, economics, finance, and insurance. However, broader variants of these distributions are clearly needed in many applications such as lifetime analysis, finance, and insurance.

In the dependability literature, most generalised Weibull distributions have been insisted to suit specific data sets better than the classic 2 or 3 parameter model. Chen revisited a two-parameter distribution. (2000) [1]. This distribution can be bathtub-shaped or have a rising failure rate function, allowing it to fit real-life data sets. Xie et.al (2002) [14] introduced modified Weibull distribution, with the $f(t)$ defined by

$$f(t;\lambda,\tau,\alpha) = \lambda\tau\left(\frac{t}{\alpha}\right)^{\tau-1} e^{\{(\frac{t}{\alpha})^{\tau}+\lambda\alpha[1-e^{(\frac{t}{\alpha})^{\tau}}]\}}, t \geq 0,$$

where $\lambda > 0$ and $\alpha > 0$ are scale parameters and $\tau > 0$ is a shape parameter. Then survival and hazard rate are given by

$$S(t;\lambda,\tau,\alpha) = e^{\left(\lambda\alpha\left[1-e^{\left(\frac{t}{\alpha}\right)^{\tau}}\right]\right)} \quad (1)$$

and

$$h(t;\lambda,\tau,\alpha) = \mu(t) = \lambda\tau\left(\frac{t}{\alpha}\right)^{\tau-1} e^{\left(\frac{t}{\alpha}\right)^{\tau}} \quad (2)$$

The hazard function is of both bathtub shaped and an increasing function.

3 Modified Weibull distribution with age dependent shape parameter

If the shape parameter τ is age - dependent (represented by r(t)), then the survival function (1) takes the form

$$S(t) = e^{\left(\lambda\alpha\left[1-e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\right]\right)} \quad (3)$$

Taking logarithm on both sides of (3) we get

$$lnS(t) = \lambda\alpha\left[1 - e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\right].$$

Further simplification gives

$$e^{\left(\frac{t}{\alpha}\right)^{r(t)}} = 1 - \frac{lnS(t)}{\lambda\alpha}$$

Taking logarithm again on both sides and simplifying further, the above equation results in

$$r(t) = \frac{ln\{ln[1 - \frac{lnS(t)}{\lambda\alpha}]\}}{ln\left(\frac{t}{\alpha}\right)} \quad (4)$$

Note that when $\alpha \to \infty$, the extended Weibull distribution has the Weibull distribution as a special and asymptotic case as discussed in [14]. The mortality function $\mu(t)$ is described by the mathematical relationship with the survival function as follows

$$\mu(t) = -\frac{dlnS(t)}{dt}. \quad (5)$$

On account of (3), we get

$$\begin{aligned}\mu(t) &= \frac{-d}{dt}\left[\lambda\alpha\left(1 - e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\right)\right] \\ &= \lambda\alpha\frac{d}{dt}\left(e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\right)\end{aligned}$$

Therefore the mortality function for the distribution is

$$\mu(t) = \lambda\alpha e^{\left(\frac{t}{\alpha}\right)^{r(t)}}\left(\frac{t}{\alpha}\right)^{r(t)}\left[\frac{r(t)}{t} + ln\left(\frac{t}{\alpha}\right)\frac{dr(t)}{dt}\right] \quad (6)$$

4

The initial concept was as follows: survival curves in their natural state exhibit (i) a quick decline in survival during the starting years of life, (ii) a somewhat stable decline, and finally (iii) an abrupt decline towards death. Surprisingly, the first behaviour matches the Weibull survival function for $r1$, while the second behaviour appears to follow the situation of $r >> 1$.

In our previous work (**??**), we gave an estimation for the shape parameter $r(t)$ in the neighbourhood of the scale parameter α which takes the form

$$\begin{aligned} r(t) &= r(\alpha) + \left[\frac{\lambda e^2 r^2(\alpha)}{2} - \frac{r^2(\alpha)}{\alpha} + \frac{r(\alpha)}{2\alpha}\right](t-\alpha) \quad (7)\\ &- \left[\frac{r(t_m)}{t_m(t_m-\alpha)\,ln(t_m/\alpha)} + \frac{\lambda e^2 r^2(\alpha)}{2(t_m-\alpha)} - \frac{r^2(\alpha)}{\alpha(t_m-\alpha)} + \frac{r(\alpha)}{2\alpha(t_m-\alpha)}\right]\frac{(t-\alpha)^2}{2!}. \end{aligned}$$

4 Properties of mortality function

4.1 Behaviour of the mortality function at the vertex

(i) At the vertex $t = \nu$, $r'(t) = 0$ and hence the mortality function of the extended Weibull model given by (6) becomes

$$\mu(\nu) = \lambda\alpha e^{\left(\frac{\nu}{\alpha}\right)^{r(\nu)}}\left[\frac{r(\nu)}{\nu}\right]$$
$$= \lambda e^{\left(\frac{\nu}{\alpha}\right)^{r(\nu)}}\left(\frac{\nu}{\alpha}\right)^{r(\nu)-1}[r(\nu)]. \quad (8)$$

As $r(\nu) > 1$ (8) gives

$$\mu(\nu) > \left(\frac{\nu}{\alpha}\right)^{r(\nu)-1}. \quad (9)$$

On the otherhand, $\nu > \alpha$ implies

$$\mu(\nu) > r(\nu). \quad (10)$$

From (9) and (10) we get

$$\mu(\nu) \geq max\left\{\left(\frac{\nu}{\alpha}\right)^{r(\nu)-1}, r(\nu)\right\}.$$

When $\nu < \alpha$, obviously (1) satisfies

$$\mu(\nu) < r(\nu).$$

(ii) Now we prove that whenever $\nu > \alpha$, $\mu(\nu) > \mu(\alpha)$ implies $r(\nu) > r(\alpha)$.

Let $\mu(\nu) > \mu(\alpha)$. Then from (8) and

$$r(\alpha) = \frac{\mu(\alpha)}{\lambda e} \quad (11)$$

we have

$$\lambda e^{\left(\frac{\nu}{\alpha}\right)^{r(\nu)}}\left(\frac{\nu}{\alpha}\right)^{r(\nu)-1}[r(\nu)] > \lambda e r(\alpha) \quad (12)$$

$$\frac{r(\nu)}{r(\alpha)} > \left(\frac{\alpha}{\nu}\right)^{r(\nu)-1} e^{1-\left(\frac{\nu}{\alpha}r(\nu)\right)} > \left(\frac{\alpha}{\nu}\right)^{r(\nu)}. \quad (13)$$

When $\nu > \alpha$

$$\left(\frac{\alpha}{\nu}\right)^{r(\nu)-1} = \left(\frac{\alpha}{\nu}\right)^{r(\nu)}\frac{\nu}{\alpha} < \frac{\nu}{\alpha}. \quad (14)$$

From (13) and (14) we get

$$\left(\frac{\alpha}{\nu}\right)^{r(\nu)} < \left(\frac{\alpha}{\nu}\right)^{r(\nu)-1} < \frac{\nu}{\alpha} = 1 + \frac{\nu-\alpha}{\alpha}. \quad (15)$$

Since

$$\frac{r(\nu)}{r(\alpha)} > \left(\frac{\alpha}{\nu}\right)^{r(\nu)}$$

and

$$\left(\frac{\alpha}{\nu}\right)^{r(\nu)} < 1 + \frac{\nu - \alpha}{\alpha},$$

it is possible to have $\frac{r(\nu)}{r(\alpha)} > 1$ which implies $r(\nu) > r(\alpha)$.

4.2 Signs of the coefficients of the age-dependent shape parameter

Weon et.al. [12] put forward that with the quadratic pattern of the agedependent parameter, the mortality pattern tends to decrease after a plateau and ultimately approach zero. The point where the mortality curve starts to decline is obtained by solving $\mu'(t) = 0$.

(i) To determine the sign of the coefficients of $r(t)$ we represent the coefficients of r, r', r'' as r_0, r_1, r_2 respectively. First we determine the sign of r_2. Differentiating (6) with respect to t and equating it to zero gives (16)

$$\lambda\alpha e^{(t/\alpha)^{r(t)}}\left[(t/\alpha)^{r(t)}\left(\frac{r(t)}{t} + lnt/\alpha\frac{dr(t)}{dt}\right)\right]^2$$
$$+\lambda\alpha e^{(t/\alpha)^{r(t)}}\left[(t/\alpha)^{r(t)}\left(\frac{tr'(t) - r(t)}{t^2}\right) + \frac{1}{t}\frac{dr(t)}{dt} + lnt/\alpha\frac{d^2r(t)}{dt^2}\right]$$
$$+\lambda\alpha e^{(t/\alpha)^{r(t)}}(t/\alpha)^{r(t)}\left[\frac{r(t)}{t} + lnt/\alpha\frac{dr(t)}{dt}\right]^2 = 0$$

It follows from (17) that

$$\lambda\alpha e^{(t/\alpha)^{r(t)}}\left[(t/\alpha)^{r(t)}\left(\frac{tr'(t) - r(t)}{t^2}\right) + \frac{1}{t}\frac{dr(t)}{dt} + lnt/\alpha\frac{d^2r(t)}{dt^2}\right] < 0, \quad t > \alpha.$$

Simplyfying further we arrive at

$$\frac{r'(t)}{t}[(t/\alpha)^{r(t)} + 1] - \frac{r(t)}{t^2} + ln\frac{t}{\alpha}r''(t) < 0, \quad t > \alpha.$$

Since $r(t) > 0$, dividing each term by $r(t)$ gives

$$\frac{r'(t)}{r(t)t}[(t/\alpha)^{r(t)} + 1] - \frac{1}{t^2} + ln\frac{t}{\alpha}\frac{r''(t)}{r(t)} < 0, \quad t > \alpha.$$

The above equation can be simplified further as

$$\frac{r'(t)}{r(t)t}[(t/\alpha)^{r(t)} + 1] + ln\frac{t}{\alpha}\frac{r''(t)}{r(t)} < \frac{1}{t^2} < \frac{1}{\alpha^2}, \quad t > \alpha.$$

That is,

$$\frac{r'(t)}{r(t)t}[(t/\alpha)^{r(t)} + 1] + ln\frac{t}{\alpha}\frac{r''(t)}{r(t)} - \frac{1}{\alpha^2} < 0, \quad t > \alpha$$

or equivalently,

$$\alpha\frac{r'(t)}{r(t)t}[(t/\alpha)^{r(t)} + 1] + \alpha ln\frac{t}{\alpha}\frac{r''(t)}{r(t)} - \frac{1}{\alpha} < 0, \quad t > \alpha$$

and taking into account $r'' = r_2$ the last equation satisfies

$$\alpha ln\frac{t}{\alpha}\frac{r_2(t)}{r(t)} - \frac{1}{\alpha} < -\alpha\frac{r'(t)}{r(t)t}[(t/\alpha)^{r(t)} + 1] < -\frac{r'(t)}{r(t)}[(t/\alpha)^{r(t)} + 1], \quad t > \alpha$$

It is well known that $S(t)$ is mathematically a monotonic decay function of age. If the change in survival probability is minimal, the preceding equation indicates a mathematical tendency ($\frac{dS(t)}{dt} \rightarrow 0$), then at young ages, the slope of the stretched exponent falls below a particular positive value ($t < \alpha$), while at older ages, it is greater than a specific negative value ($t > \alpha$). Thus the above inequality takes the form

$$ln\frac{t}{\alpha}\frac{r_2(t)}{r(t)} < \frac{1}{\alpha t ln\frac{t}{\alpha}} + \frac{1}{\alpha^2}, \quad t > \alpha$$

or,

$$\frac{r_2(t)}{r(t)} < \frac{1}{\alpha t(ln\frac{t}{\alpha})^2} + \frac{1}{\alpha^2 ln\frac{t}{\alpha}}, \quad t > \alpha$$

When t is large, the terms on the RHS of the above inequality tends to zero and since $r(t) > 0$, it follows that

$$r_2 < 0$$

(ii) Next we determine the sign of r_1. Since

$$r'(t) = r_1 + 2r_2t < 0, t > \alpha$$

we get

$$\alpha < t < \frac{-r_1}{2r_2}. \quad (17)$$

Since $r_2 < 0$, it follows from (17) that

$$r_1 > 0. \quad (18)$$

Setting $r_2 = -r_2^*,\ r_2^* > 0,\ r(t)$ takes the form

$$r(t) = r_0 + r_1t - r_2^*t^2. \quad (19)$$

Remark: Note that for $t < \alpha$,

$$r'(t) = r_1 - 2\,r_2^*t > 0$$

implies $r_1 - r_2^*t > r_2^*t >0$ and thus we have $t < \frac{r_1}{r_2^*}$.

(iii) Finally, we determine the sign of β_0.

Substitution of (19) into $r'(t)$ as $t \to \alpha$ gives

$$r_1 - 2\,r_2^*\alpha > \frac{r_0 + r_1\alpha - r_2^*\alpha^2}{2\alpha}.$$

Simplifying further, the above equation reduces to

$$r_1 - 3\,r_2^*\alpha > \frac{r_0}{\alpha}.$$

Since $r_2^* >0$, dividing each term of the above equaion by $2r_2^*$ gives

$$\frac{r_1}{2\,r_2^*} - \frac{3\alpha}{2} > \frac{r_0}{2\,r_2^*\alpha}. \quad (20)$$

Rewriting the above equation in the form

$$\frac{r_0}{2\,r_2^*\alpha} < \frac{r_1}{2\,r_2^*} - \frac{3\alpha}{2} < 0$$

it is clear that $r_0 < 0$, since $r_2^* > 0$.

Setting $r_0 = -r_0^*,\ r_0^* >0$, finally $r(t)$ takes the form

$$r(t) = -r_0^* + r_1t - r_2^*t^2. \quad (21)$$

The above simplified form of $r(t)$ is used to give a new representation for the age-dependent shape parameter.

Conclusion

The study on properties of the mortality rate function is necessary in finding the critical point and zero of the mortality function. These findings will be dealt in our future work.

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