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THEORY OF QUOTIENT MATRIX

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Abstract: Generally, when we discuss operations on matrices, we introduce addition, subtraction, scalar multiplication and even multiplication. We never discuss the concept of division of two square matrices. In this paper, I have introduced the concept of division of two square matrices under certain conditions. In fact, we have introduced the term 'Quotient Matrix' $\frac{A}{B}$ for two square matrices A and B of the same order provided AB = BA and B is a non-singular matrix. We have also established all the parallel results for 'Quotient Matrix' related to algebra of Quotient Matrices, adjoint of a Quotient matrix, inverse of a Quotient matrix and determinant of a quotient matrix.

Keywords: Quotient Matrix

INTRODUCTION

- **Why do we stop at matrix multiplication while doing algebra of matrices?**
- **Why did we not talk about matrix division?**

<u>NOTE:</u> Let us go to real number system where we learnt division. If *a* and b are two real numbers, b ≠ 0 then $\frac{a}{b}$ is defined as solution of the equations b*x* = *a* and *x*b = *a*. We know that b*x* = *x*b (by commutativity in real numbers), therefore uniqueness of *a* is preserved so the quotient $\frac{a}{b}$ is meaningful.

Now, if we consider A and B as two square matrices (of same order), B ≠ 0 and suppose $\frac{A}{B} = C$ (where C is a square matrix of same order as of A and B) then A = BC or A = CB are the consequent matrix equation BC ≠ CB (in general), therefore uniqueness of A gets violated. Hence $\frac{A}{B}$ does not make sense in case of matrices.

MATERIAL AND METHODS

Let A and B are two square matrices of order n ×n such that:

(i) AB = BA (ii) $|B| \neq 0$

Then we define quotient matrix $\frac{A}{B}$ as a matrix C of order n ×n such that $C = AB^{-1}$

NOTE: If A and B are commuting matrices (of same order) then $AB^{-1} = B^{-1}A$.

$\therefore$ C is uniquely determined.

Example 1: $A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix},\ B=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

Then AB = BA and |B| = 1 ($\neq 0$)

$\therefore \frac{A}{B}=C$ where $C = AB^{-1}$

$$C=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\alpha & \sin 2\alpha\\ -\sin 2\alpha & \cos 2\alpha\end{bmatrix}$$

Example 2: $A=\begin{bmatrix}6 & 0\\ 0 & 7\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$

Then AB = BA and |B| = 2 ($\neq 0$)

$\therefore \frac{A}{B}=C$ where $C = AB^{-1}$

$$C=\begin{bmatrix}6 & 0\\ 0 & 7\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}$$

$$=\begin{bmatrix}6 & 0\\ 0 & 7/2\end{bmatrix}$$

Algebra of Quotient Matrices

Let $\frac{A}{B}$ and $\frac{C}{D}$ are two quotient matrices (where A, B, C and D are square matrices of order *n*), then we can define.

$\frac{A}{B}+\frac{C}{D}$, $\frac{A}{B}-\frac{C}{D}$, $\alpha\frac{A}{B}$ (for α to be a scalar) and $\frac{A}{B}\cdot\frac{C}{D}$ in usual manner in which we have defined X + Y, X – Y, αX and XY. (for two suitable matrices x and y)

Some more properties which one can easily verify are:

(1) $adj\,\alpha\left(\frac{A}{B}\right)=\alpha^{n-1}\,adj.\left(\frac{A}{B}\right)$, where α is a scalar and *n* is the order of the square matrices A and B.

(2) For $n\in\mathbb{Z}^{+}$, $adj\left(\frac{A}{B}\right)^{n}=\left(adj\left(\frac{A}{B}\right)\right)^{n}$

(3) $adj\left(\frac{A}{B}.\frac{C}{D}\right)=adj\,\frac{C}{D}\,.\,adj\,\frac{A}{B}$

(where *adj* denotes adjoint)

RESULTS AND DISCUSSION

Result 1: $\left|\frac{A}{B}\right|=\frac{|A|}{|B|}$. (where |A| = determinant of matrix A)

Proof: Consider, $\left|\frac{A}{B}\right|=|AB^{-1}|=|A||B^{-1}|=\frac{|A|}{|B|}$.

Result2: For $n\in\mathbb{Z}^{+}$, we have $\left|\left(\frac{A}{B}\right)^{n}\right|=\left|\frac{A}{B}\right|^{n}$.

Proof: Consider,

$$\left|\left(\frac{A}{B}\right)^{n}\right|=|(AB^{-1})^{n}|=|A^{n}(B^{-1})^{n}| \qquad (\because AB^{-1}=B^{-1}A)$$

$$=|A^{n}|\,|(B^{-1})^{n}|=\frac{|A|^{n}}{|B|^{n}}=\left|\frac{A}{B}\right|^{n} \qquad \text{(By Prop.1)}$$

Result3: $\left|adj\left(\frac{A}{B}\right)\right| = \frac{|adj\ A|}{|adj\ B|}$.

Proof: Consider,

$$\left|adj\left(\frac{A}{B}\right)\right| = \left|adj\ AB^{-1}\right| = \left|adj\,(B^{-1}).adj\ A\right|$$

$$= \left|adj\,(B^{-1})\right|\left|\,adj\ A\right|$$

$$= \left|adj\ B\right|^{-1}\left|\,adj\ A\right|$$

$$= \frac{|adj\ A|}{|adj\ B|}$$

(**NOTE:** $|B| \neq 0 \Rightarrow |adj\ B| \neq 0$)

Result4: For two quotient matrix $\frac{A}{B}$ and $\frac{C}{D}$, we have

$$\left|adj\left(\frac{A}{B}.\frac{C}{D}\right)\right| = \frac{|adj\,(CA)|}{|adj\,(DB)|} = \frac{|adj\,(AC)|}{|adj\,(BD)|}.$$

Proof: Consider,

$$\left|adj\left(\frac{A}{B}.\frac{C}{D}\right)\right| = \left|adj\,\frac{C}{D}.adj\,\frac{A}{B}\right|$$

$$= \left|adj\,\frac{C}{D}\right|\ \left|adj\,\frac{A}{B}\right|$$

$$= \frac{|adj\ C|}{|adj\ D|}\ \frac{|adj\ A|}{|adj\ B|} \quad \text{.....(1)}$$

$$= \frac{|adj\,C\,.\ adj\ A|}{|adj\,D\,.\ adj\ B|}$$

$$= \frac{|adj\ (AC)|}{|adj\ (BD)|} = \left|\frac{adj\ (CA)}{adj\ (DB)}\right| \quad \text{By (1)}$$

Inverse of quotient matrix:

Let $\frac{A}{B}$ be any quotient matrix with $|A| \neq 0$, then we say $\frac{A}{B}$ is invertible if $\exists$ a quotient matrix $\frac{C}{D}$ such that $\frac{A}{B}\cdot\frac{C}{D} = I = \frac{C}{D}\cdot\frac{A}{B}$.

In this case, $\frac{C}{D} = \left(\frac{A}{B}\right)^{-1}$

Now, $\boxed{\left(\frac{A}{B}\right)^{-1} = (AB^{-1})^{-1} = BA^{-1}}$

Prop.1 $\left(\left(\frac{A}{B}\right)^{-1}\right)^{-1} = \frac{A}{B}$

Proof: LHS = $\left(\left(\frac{A}{B}\right)^{-1}\right)^{-1} = \left(\left(AB^{-1}\right)^{-1}\right)^{-1} = \left(BA^{-1}\right)^{-1} = AB^{-1} = \frac{A}{B}$

Prop.2 $\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^{-1} = I = \left(\frac{A}{B}\right)^{-1}\left(\frac{A}{B}\right)$

Proof: Consider,

$$\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^{-1} = \left(AB^{-1}\right)\left(AB^{-1}\right)^{-1}$$

$$= \left(AB^{-1}\right)\left(BA^{-1}\right)$$

$$= A\left(B^{-1}B\right)A^{-1} = AA^{-1} = I$$

Similarly, $\left(\frac{A}{B}\right)^{-1}\left(\frac{A}{B}\right) = \left(\left(AB^{-1}\right)^{-1}\right)\left(AB^{-1}\right)$

$$= B\left(A^{-1}A\right)B^{-1} = I$$

Result5: In addition, if A is non-singular then $\left|\left(\frac{A}{B}\right)^{-1}\right| = \frac{|B|}{|A|}$.

Proof: Consider,

$$\left|\left(\frac{A}{B}\right)^{-1}\right| = \left|\left(AB^{-1}\right)^{-1}\right| = \left|\left(B^{-1}\right)^{-1}A^{-1}\right| = |B|\,\left|A^{-1}\right| = \frac{|B|}{|A|}$$

Result6: Further, if $|C| \neq 0$, we can see $\left|\frac{\left(\frac{A}{B}\right)}{\left(\frac{C}{D}\right)}\right| = \frac{|AD|}{|BC|} = \frac{|DA|}{|CB|}$

Proof: $\left|\frac{\left(\frac{A}{B}\right)}{\left(\frac{C}{D}\right)}\right| = \left|\left(\frac{A}{B}\right)\left(\frac{C}{D}\right)^{-1}\right| = \left|\left(AB^{-1}\right)\left(CD^{-1}\right)^{-1}\right|$

$$= \left|AB^{-1}\right|\left|DC^{-1}\right|$$

$$=|A|\,|B^{-1}|\,|D|\,|C^{-1}|$$

$$=\frac{|A|}{|B|}\,\frac{|D|}{|C|}$$

$$=\frac{|AD|}{|BC|}=\frac{|DA|}{|CB|}$$

Assumptions:

(i) $|C| \neq 0 \Rightarrow \frac{|C|}{|D|} \neq 0$ as $=\left|\frac{C}{D}\right|=\frac{|C|}{|D|} \neq 0$

(ii) $\frac{A}{B}\cdot\frac{C}{D}=\frac{C}{D}\cdot\frac{A}{B}$

Prob.1: If $\left(\frac{A}{B}\right)^n = I$ for some positive integer n, then show that $\left(\frac{A}{B}\right)^{-1}$ exists.

Sol. Given: $\left(\frac{A}{B}\right)^n = I$

$\Rightarrow$ $\left|\left(\frac{A}{B}\right)^n\right| = |I|$

$\Rightarrow$ $\left|\frac{A}{B}\right|^n = 1$

$\Rightarrow$ $\left|\frac{A}{B}\right| = \pm 1$

$\Rightarrow$ $|A| \neq 0$ $\quad (\because |B| \neq 0)$

$\therefore$ $\left(\frac{A}{B}\right)^{-1}$ exists.

Prob.2: If $\left(\frac{A}{B}\right)$ is a 3×3 quotient matrix, suchthat $\left|\frac{A}{B}\right| = 4$. Find $\left|2\ adj\,\frac{A}{B}\right|$.

Sol. Consider,

$$\left|2adj\,\frac{A}{B}\right| = 8\left|adj\,\frac{A}{B}\right| = 8\left(\left|\frac{A}{B}\right|\right)^{3-1}$$

$$= 8\times\left(\left|\frac{A}{B}\right|\right)^2$$

$$= 8\times 16$$

$$= 128$$

Prob.3: $\frac{A}{B}\ adj\left(\frac{A}{B}\right) = \left|\frac{A}{B}\right| I = adj\left(\frac{A}{B}\right)\frac{A}{B}$.

Sol. Consider,

$$\frac{A}{B}\, adj\left(\frac{A}{B}\right)$$

$$=\left(AB^{-1}\right) adj\left(AB^{-1}\right)$$

$$=\left(AB^{-1}\right) adj\left(B^{-1}\right) adj\ A$$

$$=A\left(B^{-1}adj\left(B^{-1}\right)\right) adj\ A$$

$$=A\left|B^{-1}\right| I\ \ adj\ A\,(\because \mathrm{A}\cdot \mathrm{adj\ A} = |\mathrm{A}|\mathrm{I})$$

$$=\left|B^{-1}\right|\left(A\ adj\ A\right)$$

$$=\frac{|A|}{|B|}\ I$$

Similarly, $adj\left(\frac{A}{B}\right).\frac{A}{B}=\frac{|A|}{|B|}\ I$

Prob.4: If $\frac{A}{B}$ is a 3×3 matrix satisfying $\left|\frac{A}{B}\right|=1$ and $\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^{T}=I$. Prove that $\left|\frac{A}{B}-I\right|=0$.

Sol. Consider,

$$\left|\frac{A}{B}-I\right|=\left|\frac{A}{B}-\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^{T}\right|$$

$$=\left|\frac{A}{B}\left(I-\left(\frac{A}{B}\right)^{T}\right)\right|$$

$$=\left|\frac{A}{B}\right|\ \left|I^{T}-\left(\frac{A}{B}\right)^{T}\right|$$

$$=\left|\left(I-\left(\frac{A}{B}\right)\right)^{T}\right|$$

$$=\left|I-\frac{A}{B}\right| (\because |\mathrm{A}| = |A^{T}|)$$

$$=(-1)^{3}\left|\frac{A}{B}-I\right|$$

$\Rightarrow$ $2\left|\frac{A}{B}-I\right|=0$

$\Rightarrow$ $\left|\frac{A}{B}-I\right|=0$

Prob.5: If $\frac{A}{B}$ is a matrix order 2, such that $\left|2\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^T\right| = 16$ and $|B| = 1$. Find | A |.

Sol. Given,

$$\left|2\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^T\right| = 16$$

$$\Rightarrow \quad 4\left|\frac{A}{B}\right|\left|\left(\frac{A}{B}\right)^T\right| = 16$$

$$\Rightarrow \quad \left|\frac{A}{B}\right|^2 = 4 \quad \left(\because \left|\frac{A}{B}\right| = \left|\left(\frac{A}{B}\right)^T\right|\right)$$

$$\Rightarrow \quad \left|\frac{A}{B}\right| = \pm 2$$

$$\Rightarrow \quad \frac{|A|}{|B|} = \pm 2$$

$$\Rightarrow \quad |A| = \pm 2$$

Prob.6: If $\frac{A}{B}$ is a 2×2non-singular matrix, show that $adj\left(\frac{A}{B}\right) = adj\left(\frac{A}{B}\right)^{-1}$. Find $\left|\frac{A}{B}\right|$.

Sol. Given,

$$\left|adj\left(\frac{A}{B}\right)\right| = \left|adj\left(\frac{A}{B}\right)^{-1}\right|$$

$$\Rightarrow \quad \left|adj\left(\frac{A}{B}\right)\right| = \left|\left(adj\left(\frac{A}{B}\right)\right)\right|^{-1} = \frac{1}{\left|adj\left(\frac{A}{B}\right)\right|}$$

$$\Rightarrow \quad \left|adj\left(\frac{A}{B}\right)\right|^2 = 1$$

$$\Rightarrow \quad \left|adj\left(\frac{A}{B}\right)\right| = \pm 1$$

$$\Rightarrow \quad \left|\frac{A}{B}\right| = \pm 1 \left(\because \left|adj\frac{A}{B}\right| = \left|\frac{A}{B}\right|^{n-1}\right)$$

$$\Rightarrow \quad \left|\frac{A}{B}\right| = \pm 1$$

Prob.7: Let $\frac{A}{B}$ and $\frac{C}{D}$ are two non-singular matrixof order 2×2, such that $|B| = 2 = |D|$, $\left(\frac{A}{B}\right)^2 = \frac{A}{B}\cdot\frac{C}{D}$ and $\left(\frac{C}{D}\right)^2 = \left(\frac{A}{B}\cdot\frac{C}{D}\right)^{-1}$. Find | A | and | C |.

Sol. Given,

$$\left(\frac{A}{B}\right)^2 = \frac{A}{B} \cdot \frac{C}{D}$$

$$\Rightarrow \quad \left|\frac{A}{B}\right|^2 = \left|\frac{A}{B}\right| \left|\frac{C}{D}\right|$$

$$\Rightarrow \quad \left|\frac{A}{B}\right| = \left|\frac{C}{D}\right|$$

Also, $\left|\left(\frac{C}{D}\right)\right|^2 = \left|\frac{A}{B} \cdot \frac{C}{D}\right|^{-1}$

$$\Rightarrow \quad \left|\frac{C}{D}\right|^2 = \frac{1}{\left|\frac{A}{B}\right| \left|\frac{C}{D}\right|}$$

$$\Rightarrow \quad \left|\frac{C}{D}\right|^4 = 1 \qquad \left(\because \left|\frac{A}{B}\right| = \left|\frac{C}{D}\right|\right)$$

$$\Rightarrow \quad \left|\frac{C}{D}\right| = \pm 1$$

$$\therefore \quad \left|\frac{A}{B}\right| = \left|\frac{C}{D}\right| = \pm 1$$

Also, $\left|\frac{A}{B}\right| = \pm 1 \quad \Rightarrow \frac{|A|}{|B|} = \pm 1$

$$\Rightarrow \quad |A| = \pm |B| \qquad \therefore \ |A| = \pm 2$$

Similarly, $|C| = \pm 2$

Prob.8: $\left(\left(\frac{A}{B}\right)^T\right)^{-1} = \left(\left(\frac{A}{B}\right)^{-1}\right)^T$.

Sol. Consider,

$$\left(\left(\frac{A}{B}\right)^{-1}\right)^T = \left(\left(AB^{-1}\right)^{-1}\right)^T$$

$$= \left(\left(AB^{-1}\right)^T\right)^{-1}$$

$$= \left(\left(\frac{A}{B}\right)^T\right)^{-1}$$

Prob.9: For any quotient matrix $\frac{A}{B}$ (with $|A| \neq 0$) and a non-zero scalar α, we have:

$$\left(\alpha \frac{A}{B}\right)^{-1} = \frac{1}{\alpha}\left(\frac{A}{B}\right)^{-1}.$$ (Verify yourself)

Prob.10: For two quotient matrix $\frac{A}{B}$ and $\frac{C}{D}$ (with $|A| \neq 0$, $|C| \neq 0$) (for which $\frac{A}{B} \cdot \frac{C}{D}$ is defined), we have:

$$\left(\frac{A}{B} \cdot \frac{C}{D}\right)^{-1} = \left(\frac{C}{D}\right)^{-1}\left(\frac{A}{B}\right)^{-1}.$$ (Verify yourself)

CONCLUSION

The concept of division of two square matrices can be defined under certain assumed conditions. In fact, we can talk about the quotient matrix '$\frac{A}{B}$' and verify that all the parallel results related to algebra of matrices, adjoint of a matrix, inverse of a matrix and determinant of a matrix hold true in case of quotient matrix.

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