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ON THE APPLICATIONS OF ALEPH(ℵ)-FUNCTION IN A SLIGHTLY DIFFERENT TYPE-1 BETA DENSITY MODEL

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Abstract

In the present paper, the author has studied about the structures which are the products and ratios of statistically independently distributed positive real scalar random variables.The author has derived the exact density of a slightly different Type-1 beta density by the Mellin Transform and Hankel Tranform of the unknown density and afterthat the unknown density has been derived in terms of Aleph functions by taking the inverse Mellin transform and Inverse Hankel Transform .A more general structure of Type-1 beta density has also been discussed. Some special cases in terms of H -function are also given.

Key words: Type-1 Beta density, Aleph Function, I -function, H -function, Mellin Transform, Inverse Mellin Transform, Hankel Transform, Inverse Hankel Transform.

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1. Introduction

The I -function introduced by Saxena [4] will be represented and defined in slightly different manner as follows:

$$I(z)=I^{m,n}_{p_i,q_i:R}(z)=I^{m,n}_{p_i,q_i:R}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\theta(s)z^{-s}ds \quad (1)$$

Where $i=\sqrt{-1},\ z\neq 0$ and $z^{-s}=\exp[-\sin|z|+i\arg z]$ where $|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not the principal value. Here

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{R}\left\{\prod_{j=m+1}^{q}\Gamma\left(1-b_{ji}-B_{ji}s\right)\prod_{j=n+1}^{p}\Gamma\left(a_{ji}+A_{ji}s\right)\right\}} \quad (2)$$

For $R=1$, the I -function reduces to the H -function.

The $\aleph$ - function introduced by Suland et.al. [6] defined and represented in the following form:

$$\aleph[z]=\aleph_{p_i,q_i;\tau_i;r}^{m,n}[z]=\aleph_{p_i,qi;\tau_i;r}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_L \theta(s)z^s ds \qquad (3)$$

Where $\omega=\sqrt{-1}$;

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s\right\}} \qquad (4)$$

We shall use the following notation:

$$A^*=\left(a_j,\alpha_j\right)_{1,n},[\tau_i\left(a_{ji},\alpha_{ji}\right)]_{n+1,p_i},B^*=\left(b_j,\beta_j\right)_{1,m},[\tau_i\left(b_{ji},\beta_{ji}\right)]_{m+1,q_i}$$

The Mellin transform of $f(x)$ denoted by $M\{f(x);s\}$ or $F(s)$ is given by

$$M\{f(x);s\}=\int_0^\infty x^{s-1}f(x)dx \qquad (5)$$

The Hankel transform of $f(x)$ denoted by $H_v\{f(x);p\}$ or $F_v(p)$ is given by

$$H_v\{f(x);p\}=\int_0^\infty x\,J_v(px)\,f(x)dx \qquad (6)$$

A real scalar random variable x is said to have a real type-1 generalized beta distribution,if the density is of the following form ([1], p.121, eq. (4.8)):

$$f(x)=\begin{cases}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1};\ 0<x<1,\alpha>0,\beta>0\\0,elsewhere\end{cases} \qquad (7)$$

Where the parameters α and β are real. The following discussion holds even when α and β are complex quantities. In this case, the conditions become $\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0$ where Re(.)means the real part of (.).

2. General structures

A real scalar random variable x is said to have a real generalized type-1 generalized beta distribution,if the density is of the following form:

$$f(x)=\begin{cases}\frac{\Gamma\left(\frac{\alpha+tm}{\gamma}+\beta\right)A^{\frac{\alpha+tm}{\gamma}}}{\Gamma\left(\frac{\alpha+tm}{\gamma}\right)\Gamma(\beta)}x^{\alpha-1}(1-Ax)^{\beta-1}\,{}_2F_1(a,b;c,w;\mu;px^t)\\0,elsewhere\end{cases} \qquad (8)$$

For $0 < x < A^{\frac{-1}{\gamma}}, \alpha > 0, \beta > 0, A > 0, a > 0, b > 0, c > 0, 1 - Ax^{\gamma} > 0$.

Where the parameters α and β are real. The following discussion holds even when α and β are complex quantities. In this case, the conditions become $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$ where Re(.)means the real part of (.).

The h^{th} -moment of x, when x has the density in (8), is given by

$$E\left(x^{h}\right) = \frac{\Gamma\left(\frac{\alpha + tm + h}{\gamma}\right)\Gamma\left(\frac{\alpha + tm}{\gamma} + \beta\right)}{A^{\frac{h}{\gamma}}\Gamma\left(\frac{\alpha + tm}{\gamma}\right)\Gamma\left(\frac{\alpha + h}{\gamma} + \beta\right)} \qquad (9)$$

For $\mathrm{Re}(\alpha + tm + h) > 0, \mathrm{Re}(\alpha + tm + \beta) > 0, \gamma > 0$.

When α and h are real, the moments can exist for some values of h also such that $\alpha + h > 0$.

The Mellin transform of $f(x)$ is obtained from (8) as:

$$M\left\{f(x)\right\} = \frac{\Gamma\left(\frac{\alpha + tm + s - 1}{\gamma}\right)\Gamma\left(\frac{\alpha + tm}{\gamma} + \beta\right)}{A^{\frac{s-1}{\gamma}}\Gamma\left(\frac{\alpha + tm}{\gamma}\right)\Gamma\left(\frac{\alpha + s - 1}{\gamma} + \beta\right)} \qquad (10)$$

For $\mathrm{Re}(\alpha + tm + s - 1) > 0, \mathrm{Re}(\alpha + tm + \beta) > 0, s = v + 2r + 2 > 0, \gamma > 0$.

The unknown density $f(x)$ is obtained in terms of $\aleph$ -function by taking the inverse Mellin transform of (10). That is

$$f(x) = \frac{A^{\frac{1}{\gamma}}\Gamma\left(\frac{\alpha + tm}{\gamma} + \beta\right)}{\Gamma\left(\frac{\alpha + tm}{\gamma}\right)} \aleph^{1,0}_{1,1:\tau_i:R}\left[A^{\frac{1}{\gamma}}x \left| \begin{matrix} \left(\frac{\alpha-1}{\gamma} + \beta, \frac{1}{\gamma}\right) \\ \left(\frac{\alpha+tm-1}{\gamma} + \beta, \frac{1}{\gamma}\right) \end{matrix} \right.\right] \qquad (11)$$

The Hankel transform of $f(x)$ is obtained from (9) as:

$$H\left\{f(x)\right\} = J_{v}(p)\frac{\Gamma\left(\frac{\alpha + tm + s - 1}{\gamma}\right)\Gamma\left(\frac{\alpha + tm}{\gamma} + \beta\right)}{A^{\frac{s-1}{\gamma}}\Gamma\left(\frac{\alpha + tm}{\gamma}\right)\Gamma\left(\frac{\alpha + s - 1}{\gamma} + \beta\right)} \qquad (12)$$

For $\mathrm{Re}(\alpha + tm + s - 1) > 0, \mathrm{Re}(\alpha + tm) > 0, s = v + 2r + 2 > 0, \gamma > 0$.

The unknown density $f(x)$ is obtained in terms of $\aleph$ -function by taking the inverse Hankel transform of (12). That is

$$f(x)=J_{v}(p)\frac{A^{\frac{1}{\gamma}}\Gamma\left(\frac{\alpha+tm}{\gamma}+\beta\right)}{\Gamma\left(\frac{\alpha+tm}{\gamma}\right)}\aleph^{1,0}_{1,1:\tau_i:R}\left[A^{\frac{1}{\gamma}}x\left|\begin{matrix}\left(\frac{\alpha-1}{\gamma}+\beta,\frac{1}{\gamma}\right)\\ \left(\frac{\alpha+tm-1}{\gamma}+\beta,\frac{1}{\gamma}\right)\end{matrix}\right.\right] \qquad (13)$$

Consider a set of real scalar random variables $x_1,\ldots,x_k$, mutually independently distributed, where x_j has the density in (8) with the parameters $\alpha_j,\beta_j;\ j=1,\ldots,k$ and consider the product

$$u=x_1x_2\ldots x_k \qquad (14)$$

In the standard terminology in statistical literature, the h^{th} moment of u, when u has the density in (8), is given by

$$E(u)=\frac{\Gamma\left(\frac{\alpha_j+tm+h}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j+tm}{\gamma}+\beta_j\right)}{A^{\frac{h}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j+h}{\gamma_j}+\beta_j\right)} \qquad (15)$$

For $\mathrm{Re}(\alpha_j+tm+h)>0,\gamma_j>0;\ j=1,\ldots,k$

Then the Mellin transform of $g(u)$ of u is obtained from the property of the statistical independent and is given by

$$M\left[g(u)\right]=M\left[x_1^{s-1}\right]\ldots M\left[x_k^{s-1}\right] \qquad (16)$$

$$M\left[g(u)\right]=\prod_{j=1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm+s-1}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j+tm}{\gamma}+\beta_j\right)}{A^{\frac{s-1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j+s-1}{\gamma_j}+\beta_j\right)} \qquad (17)$$

For $\mathrm{Re}(\alpha_j+tm+s-1)>0,\gamma_j>0, s=v+2r+2$

The unknown density $f(x)$ is obtained in terms of $\aleph$-function by taking the inverse Mellin transform of (17). That is

$$g(u)=\prod_{j=1}^{k}\frac{A^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\aleph^{k,0}_{k,k\tau_i:R}\left[\prod_{j=1}^{k}A^{\frac{1}{\gamma_j}}u\left|\begin{matrix}\left(\frac{\alpha_j-1}{\gamma}+\beta_j,\frac{1}{\gamma_j}\right)\\ \left(\frac{\alpha_j+tm-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.\right] \qquad (18)$$

Then the Hankel transform of $g(u)$ of u is obtained from the property of the statistical independent and is given by:

$$H[g(u)]=H\left[x_1J_v(px_1)\right]H\left[x_2J_v(px_2)\right]\ldots H\left[x_kJ_v(px_k)\right] \qquad (19)$$

$$H\left\{g(u)\right\} = J_v(p)\prod_{j=1}^{k}\frac{\Gamma\left(\frac{\alpha_j + tm + s - 1}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j + tm}{\gamma} + \beta_j\right)}{A^{\frac{s-1}{\gamma_j}}\Gamma\left(\frac{\alpha_j + tm}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j + s - 1}{\gamma_j} + \beta_j\right)} \quad (20)$$

For $\text{Re}(\alpha_j + tm + s - 1) > 0, \gamma_j > 0, s = v + 2r + 2 > 0$

The unknown density $f(x)$ is obtained in terms of $\aleph$-function by taking the inverse Hankel transform of (20). That is

$$g(u) = J_v(p)\prod_{j=1}^{k}\frac{A^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j + tm}{\gamma_j} + \beta_j\right)}{\Gamma\left(\frac{\alpha_j + tm}{\gamma_j}\right)}\aleph_{k,k:\tau_i:R}^{k,0}\left[\prod_{j=1}^{k}A^{\frac{1}{\gamma_j}}u\left|\begin{matrix}\left(\frac{\alpha_j - 1}{\gamma} + \beta_j, \frac{1}{\gamma_j}\right)\\ \left(\frac{\alpha_j + tm - 1}{\gamma_j} + \beta_j, \frac{1}{\gamma_j}\right)\end{matrix}\right.\right] \quad (21)$$

If we consider more general structures in the same category. For example, consider the structure

$$u_1 = x_1^{\delta_1}x_2^{\delta_2}\ldots x_k^{\delta_k}, \delta_j > 0, j = 1, \ldots, k \quad (22)$$

Where $x_1, \ldots, x_k$ are mutually independently distributed as in (8).

Then the Mellin transform of $g(u_1)$ of u_1 is given as

$$M\left[g(u_1)\right] = M\left[x_1^{\gamma_1(s-1)}\right]\ldots M\left[x_k^{\gamma_k(s-1)}\right] \quad (23)$$

$$M\left[g(u_1)\right] = \prod_{j=1}^{k}\frac{\Gamma\left(\frac{\alpha_j + tm + \delta_j(s-1)}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j + tm}{\gamma_j} + \beta_j\right)}{A^{\frac{\delta_j(s-1)}{\gamma_j}}\Gamma\left(\frac{\alpha_j + tm}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j + \delta_j(s-1)}{\gamma_j} + \beta_j\right)} \quad (24)$$

For $\text{Re}(\alpha_j + tm + s - 1) > 0, s = v + 2r + 2, \delta_j > 0$.

The unknown density $g(u_1)$ is obtained in terms of $\aleph$-function by taking the inverse Mellin transform of (24). That is

$$g(u_1) = \prod_{j=1}^{k}\frac{A^{\frac{\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j + tm}{\gamma_j} + \beta_j\right)}{\Gamma\left(\frac{\alpha_j + tm}{\gamma_j}\right)}\aleph_{k,k:\tau_i:R}^{k,0}\left[\prod_{j=1}^{k}A^{\frac{\delta_j}{\gamma_j}}u_1\left|\begin{matrix}\left(\frac{\alpha_j - \delta_j}{\gamma} + \beta_j, \frac{\delta_j}{\gamma_j}\right)\\ \left(\frac{\alpha_j + tm - \delta_j}{\gamma_j} + \beta_j, \frac{\delta_j}{\gamma_j}\right)\end{matrix}\right.\right] \quad (25)$$

For $\text{Re}(\alpha_j + tm + s - 1) > 0, s = v + 2r + 2, \delta_j > 0$.

The Hankel transform of $g(u_1)$ of u_1 is obtained from the property of the statistical independent and is given by:

$$H[g(u_1)] = H\left[x_1^{\delta_1}J_v(px_1^{\delta_1})\right]H\left[x_2^{\delta_2}J_v(px_2^{\delta_2})\right]\ldots H\left[x_k^{\delta_k}J_v(px_k^{\delta_k})\right] \quad (26)$$

$$H\{g(u_1)\} = J_v(p)\prod_{j=1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm+\delta_j(s-1)}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{A^{\frac{\delta_j(s-1)}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)\Gamma\left(\frac{\alpha_j+\delta_j(s-1)}{\gamma_j}+\beta_j\right)} \quad (27)$$

For $\mathrm{Re}(\alpha_j+tm+s-1)>0, s=v+2r+2>0, \delta_j>0$.

The unknown density $g(u_1)$ is obtained in terms of $\aleph$-function by taking the inverse Hankel transform of (27). That is

$$g(u_1) = J_v(p)\prod_{j=1}^{k}\frac{A^{\frac{\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\aleph_{k,k:\tau_i:R}^{k,0}\left[\prod_{j=1}^{k}A^{\frac{\delta_j}{\gamma_j}}u_1\left|\begin{matrix}\left(\frac{\alpha_j-\delta_j}{\gamma}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-\delta_j}{\gamma_j}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\end{matrix}\right.\right] \quad (28)$$

For $\mathrm{Re}(\alpha_j+tm+s-1)>0, s=v+2r+2>0, \delta_j>0$.

A More General Structure

We can consider more general structures. Let

$$w=\frac{x_1,x_2,\ldots,x_r}{x_{r+1},\ldots,x_k} \quad (29)$$

Where $x_1,\ldots,x_k$, mutually independently distributed real random variables having the density in (8) with x_j having parameters $\alpha_j,\beta_j; j=1,\ldots,k$.

Then the Mellin transform of $g(w)$ is given by

$$M[g(w)] = M\left[x_1^{s-1}\right]\ldots M\left[x_r^{s-1}\right]M\left[x_{r+1}^{-(s-1)}\right]\ldots M\left[x_k^{-(s-1)}\right] \quad (30)$$

$$M[g(w)] = \prod_{j=1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{A^{\frac{-1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{\prod_{j=1}^{r}\frac{\Gamma\left(\frac{\alpha_j+tm+s-1}{\gamma_j}\right)}{A^{\frac{s}{\gamma_j}}\Gamma\left(\frac{\alpha_j+s-1}{\gamma_j}+\beta_j\right)}\right\}$$

$$\left\{\prod_{j=r+1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm-(s-1)}{\gamma_j}\right)}{A^{\frac{-s}{\gamma_j}}\Gamma\left(\frac{\alpha_j-(s-1)}{\gamma_j}+\beta_j\right)}\right\} \quad (31)$$

For $\mathrm{Re}(\alpha_j+tm\pm(s-1))>0, s=v+2r+2>0$

The unknown density $g(w)$ is obtained in terms of $\aleph$-function by taking the inverse Mellin transform of (31). That is

$$g(w)=\prod_{j=1}^{k}\frac{A^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{\aleph_{r,r:\tau_i:R}^{r,0}\left[\prod_{j=1}^{k}A^{\frac{1}{\gamma_j}}w\left|\begin{matrix}\left(\frac{\alpha_j-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.\right]\right\}$$

$$\left\{\aleph_{k-r,k-r:\tau_i:R}^{0,k-r}\left[\prod_{j=r+1}^{k}A^{\frac{1}{\gamma_j}}w\left|\begin{matrix}\left(1-\frac{\alpha_j+tm+1}{\gamma_j},\frac{1}{\gamma_j}\right)\\\left(1-\frac{\alpha_j+1}{\gamma_j}-\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.;j=r+1,\ldots,k\right]\right\} \quad (32)$$

Then the Hankel transform of $g(w)$ is given as:

$$H[g(w)]=H\left[x_1J_v(px_1)\right]\ldots H\left[x_rJ_v(px_r)\right]H\left[x_{r+1}^{-1}J_v\left(px_{r+1}^{-1}\right)\right]\ldots H\left[x_k^{-1}J_v\left(px_k^{-1}\right)\right] \quad (33)$$

$$H\left\{g(w)\right\}=J_v(p)\prod_{j=1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{A^{\frac{-1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{\prod_{j=1}^{r}\frac{\Gamma\left(\frac{\alpha_j+tm+s-1}{\gamma_j}\right)}{A^{\frac{s}{\gamma_j}}\Gamma\left(\frac{\alpha_j+s-1}{\gamma_j}+\beta_j\right)}\right\}$$

$$\left\{\prod_{j=r+1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm-(s-1)}{\gamma_j}\right)}{A^{\frac{-s}{\gamma_j}}\Gamma\left(\frac{\alpha_j-(s-1)}{\gamma_j}+\beta_j\right)}\right\} \quad (34)$$

For $\mathrm{Re}(\alpha_j+tm\pm(s-1))>0, s=v+2r+2>0$

The unknown density $g(w)$ is obtained in terms of $\aleph$-function by taking the inverse Hankel transform of (34). That is

$$g(w)=J_v(p)\prod_{j=1}^{k}\frac{A^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{\aleph_{r,r:\tau_i:R}^{r,0}\left[\prod_{j=1}^{k}A^{\frac{1}{\gamma_j}}w\left|\begin{matrix}\left(\frac{\alpha_j-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.\right]\right\}$$

$$\left\{\aleph_{k-r,k-r:\tau_i;R}^{0,k-r}\left[\prod_{j=r+1}^{k}A^{\frac{1}{\gamma_j}}w\left|\begin{matrix}\left(1-\frac{\alpha_j+tm+1}{\gamma_j},\frac{1}{\gamma_j}\right)\\\left(1-\frac{\alpha_j+1}{\gamma_j}-\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.;j=r+1,\ldots,k\right]\right\} \quad (35)$$

We can consider more general structures in the same category. For example, consider the structure

$$w_1=\frac{x_1^{\delta_1},\ldots,x_r^{\delta_r}}{x_{r+1}^{\delta_{r+1}},\ldots,x_k^{\delta_k}} \quad (36)$$

Where $x_1,\ldots,x_k$, mutually independently distributed real random variables having the density in (8) with x_j having parameters $\alpha_j,\beta_j; j=1,\ldots,k$.

Then the Mellin transform of $g(w_1)$ is given by

$$M\left[g(w_1)\right]=M\left[x_1^{\delta_1(s-1)}\right]...M\left[x_r^{\delta_r(s-1)}\right]M\left[x_{r+1}^{-\delta_{r+1}(s-1)}\right]...M\left[x_k^{-\delta_k(s-1)}\right] \quad (37)$$

$$M\left[g(w_1)\right]=\prod_{j=1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{A^{\frac{-\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{\prod_{j=1}^{r}\frac{\Gamma\left(\frac{\alpha_j+tm+\delta_j(s-1)}{\gamma_j}\right)}{A^{\frac{\delta_j s}{\gamma_j}}\Gamma\left(\frac{\alpha_j+\delta_j(s-1)}{\gamma_j}+\beta_j\right)}\right\}$$

$$\left\{\prod_{j=r+1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm-\delta_j(s-1)}{\gamma_j}\right)}{A^{\frac{-s\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j-\delta_j(s-1)}{\gamma_j}+\beta_j\right)}\right\} \quad (38)$$

For $\operatorname{Re}(\alpha_j+tm\pm(s-1))>0, s=v+2r+2>0,\delta_j>0$.

The unknown density $g(w_1)$ is obtained in terms of $\aleph$-function by taking the inverse Mellin transform of (38). That is

$$g(w_1)=\prod_{j=1}^{k}\frac{A^{\frac{\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{\aleph^{r,0}_{r,r:\tau_i:R}\left[\prod_{j=1}^{r}A^{\frac{\delta_j}{\gamma_j}}w_1\left|\begin{matrix}\left(\frac{\alpha_j-\delta_j}{\gamma_j}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-\delta_j}{\gamma_j}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\end{matrix}\right.\right]\right\}$$

$$\left\{\aleph^{0,k-r}_{k-r,k-r:\tau_i:R}\left[\prod_{j=r+1}^{k}A^{\frac{-\delta_j}{\gamma_j}}w_1\left|\begin{matrix}\left(1-\frac{\alpha_j+tm+\delta_j}{\gamma_j},\frac{\delta_j}{\gamma_j}\right)\\\left(1-\frac{\alpha_j+\delta_j}{\gamma_j}-\beta_j,\frac{\delta_j}{\gamma_j}\right)\end{matrix}\right.;j=r+1,...,k\right]\right\} \quad (39)$$

For $\operatorname{Re}(\alpha_j+tm\pm(s-1))>0, s=v+2r+2>0,\delta_j>0$.

Then the Hankel transform of $g(w_1)$ is given as:

$$H[g(w_1)]=H\left[x_1^{\delta_1}J_v(px_1^{\delta_1})\right]...H\left[x_r^{\delta_r}J_v(px_r^{\delta_r})\right]$$

$$H\left[x_{r+1}^{-\delta_{r+1}}J_v\left(px_{r+1}^{-\delta_{r+1}}\right)\right]...H\left[x_k^{-\delta_k}J_v\left(px_k^{-\delta_k}\right)\right] \quad (40)$$

$$H\left\{g(w_1)\right\}=J_v(p)\prod_{j=1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{A^{\frac{-\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{\prod_{j=1}^{r}\frac{\Gamma\left(\frac{\alpha_j+tm+\delta_j(s-1)}{\gamma_j}\right)}{A^{\frac{\delta_j s}{\gamma_j}}\Gamma\left(\frac{\alpha_j+\delta_j(s-1)}{\gamma_j}+\beta_j\right)}\right\}$$

$$\left\{\prod_{j=r+1}^{k}\frac{\Gamma\left(\frac{\alpha_j+tm-\delta_j(s-1)}{\gamma_j}\right)}{A^{\frac{-s\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j-\delta_j(s-1)}{\gamma_j}+\beta_j\right)}\right\} \quad (41)$$

For $\text{Re}(\alpha_j+tm\pm(s-1))>0,\delta_j>0,s=v+2r+2>0$

The unknown density $g(w_1)$ is obtained in terms of $\aleph$-function by taking the inverse Hankel transform of (41). That is

$$g(w_1)=J_v(p)\prod_{j=1}^{k}\frac{A^{\frac{\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{\aleph_{r,r:\tau_i:R}^{r,0}\left[\prod_{j=1}^{r}A^{\frac{\delta_j}{\gamma_j}}w_1\left|\begin{matrix}\left(\frac{\alpha_j-\delta_j}{\gamma_j}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-\delta_j}{\gamma_j}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\end{matrix}\right.\right]\right\}$$

$$\left\{\aleph_{k-r,k-r:\tau_i:R}^{0,k-r}\left[\prod_{j=r+1}^{k}A^{\frac{-\delta_j}{\gamma_j}}w_1\left|\begin{matrix}\left(1-\frac{\alpha_j+tm+\delta_j}{\gamma_j},\frac{\delta_j}{\gamma_j}\right)\\\left(1-\frac{\alpha_j+\delta_j}{\gamma_j}-\beta_j,\frac{\delta_j}{\gamma_j}\right)\end{matrix}\right.;j=r+1,\dots,k\right]\right\} \quad (42)$$

For $\text{Re}(\alpha_j+tm\pm(s-1))>0,\delta_j>0,s=v+2r+2>0$.

3. Special Cases

If we take $\tau_i=1,R=1$ in (11),the unknown density $f(x)$ is obtained in terms of H-function . That is

$$f(x)=\frac{A^{\frac{1}{\gamma}}\Gamma\left(\frac{\alpha+tm}{\gamma}+\beta\right)}{\Gamma\left(\frac{\alpha+tm}{\gamma}\right)}H_{1,1}^{1,0}\left[A^{\frac{1}{\gamma}}x\left|\begin{matrix}\left(\frac{\alpha-1}{\gamma}+\beta,\frac{1}{\gamma}\right)\\\left(\frac{\alpha+tm-1}{\gamma}+\beta,\frac{1}{\gamma}\right)\end{matrix}\right.\right] \quad (43)$$

For $\gamma_j=1; j=1,\dots,k$, the H-function reduces to the G-function.

If we take $R=1=\tau_i$ in (13), the unknown density $f(x)$ is obtained in terms of H-function . That is

$$f(x)=J_v(p)\frac{A^{\frac{1}{\gamma}}\Gamma\left(\frac{\alpha+tm}{\gamma}+\beta\right)}{\Gamma\left(\frac{\alpha+tm}{\gamma}\right)}H_{1,1}^{1,0}\left[A^{\frac{1}{\gamma}}x\left|\begin{matrix}\left(\frac{\alpha-1}{\gamma}+\beta,\frac{1}{\gamma}\right)\\\left(\frac{\alpha+tm-1}{\gamma}+\beta,\frac{1}{\gamma}\right)\end{matrix}\right.\right] \quad (44)$$

For $\gamma_j=1; j=1,\dots,k$, the H-function reduces to the G-function.

If we take $R=1=\tau_i$ in (18), the unknown density $g(u)$ is obtained in terms of H-function . That is

$$g(u)=\prod_{j=1}^{k}\frac{A^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}H_{k,k}^{k,0}\left[\prod_{j=1}^{k}A^{\frac{1}{\gamma_j}}u\left|\begin{matrix}\left(\frac{\alpha_j-1}{\gamma}+\beta_j,\frac{1}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.\right] \tag{45}$$

For $\gamma_j=1;\ j=1,\ldots,k$, the H-function reduces to the G-function.

If we take $R=1=\tau_i$ in (21), the unknown density $g(u)$ is obtained in terms of H-function . That is

$$g(u)=J_v(p)\prod_{j=1}^{k}\frac{A^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}H_{k,k}^{k,0}\left[\prod_{j=1}^{k}A^{\frac{1}{\gamma_j}}u\left|\begin{matrix}\left(\frac{\alpha_j-1}{\gamma}+\beta_j,\frac{1}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.\right] \tag{46}$$

For $\gamma_j=1;\ j=1,\ldots,k$, the H-function reduces to the G-function.

If we take $R=1=\tau_i$ in (25), the unknown density $g(u_1)$ is obtained in terms of H-function . That is

$$g(u_1)=\prod_{j=1}^{k}\frac{A^{\frac{\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}H_{k,k}^{k,0}\left[\prod_{j=1}^{k}A^{\frac{\delta_j}{\gamma_j}}u_1\left|\begin{matrix}\left(\frac{\alpha_j-\delta_j}{\gamma}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-\delta_j}{\gamma_j}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\end{matrix}\right.\right] \tag{47}$$

For $\gamma_j=1=\delta_j;\ j=1,\ldots,k$, the H-function reduces to the G-function.

If we take $R=1=\tau_i$ in (28), the unknown density $g(u_1)$ is obtained in terms of H-function . That is

$$g(u_1)=J_v(p)\prod_{j=1}^{k}\frac{A^{\frac{\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}H_{k,k}^{k,0}\left[\prod_{j=1}^{k}A^{\frac{\delta_j}{\gamma_j}}u_1\left|\begin{matrix}\left(\frac{\alpha_j-\delta_j}{\gamma_j}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-\delta_j}{\gamma_j}+\beta_j,\frac{\delta_j}{\gamma_j}\right)\end{matrix}\right.\right] \tag{48}$$

For $\gamma_j=1=\delta_j;\ j=1,\ldots,k$, the H-function reduces to the G-function.

If we take $R=1=\tau_i$ in (32), the unknown density $g(w)$ is obtained in terms of H-function . That is

$$g(w)=\prod_{j=1}^{k}\frac{A^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+tm}{\gamma_j}\right)}\left\{H_{r,r}^{r,0}\left[\prod_{j=1}^{k}A^{\frac{1}{\gamma_j}}w\left|\begin{matrix}\left(\frac{\alpha_j-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\\\left(\frac{\alpha_j+tm-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.\right]\right\}$$

$$\left\{H_{k-r,k-r}^{0,k-r}\left[\prod_{j=r+1}^{k}A^{\frac{1}{\gamma_j}}w\left|\begin{matrix}\left(1-\frac{\alpha_j+tm+1}{\gamma_j},\frac{1}{\gamma_j}\right)\\\left(1-\frac{\alpha_j+1}{\gamma_j}-\beta_j,\frac{1}{\gamma_j}\right)\end{matrix}\right.;\ j=r+1,\ldots,k\right]\right\} \tag{49}$$

For $\gamma_j = 1;\ j = 1,\ldots,k$, the H-function reduces to the G-function.

If we take $R = 1 = \tau_i$ in (35), the unknown density $g(w)$ is obtained in terms of H-function . That is

$$g(w) = J_v(p)\prod_{j=1}^{k}\frac{A^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j + tm}{\gamma_j} + \beta_j\right)}{\Gamma\left(\frac{\alpha_j + tm}{\gamma_j}\right)}\left\{H_{r,r}^{r,0}\left[\prod_{j=1}^{k} A^{\frac{1}{\gamma_j}} w \left| \begin{matrix} \left(\frac{\alpha_j - 1}{\gamma_j} + \beta_j, \frac{1}{\gamma_j}\right) \\ \left(\frac{\alpha_j + tm - 1}{\gamma_j} + \beta_j, \frac{1}{\gamma_j}\right) \end{matrix}\right.\right]\right\}$$

$$\left\{H_{k-r,k-r}^{0,k-r}\left[\prod_{j=r+1}^{k} A^{\frac{1}{\gamma_j}} w \left| \begin{matrix} \left(1 - \frac{\alpha_j + tm + 1}{\gamma_j}, \frac{1}{\gamma_j}\right) \\ \left(1 - \frac{\alpha_j + 1}{\gamma_j} - \beta_j, \frac{1}{\gamma_j}\right) \end{matrix}\right.; j = r+1,\ldots,k\right]\right\} \tag{50}$$

For $\gamma_j = 1;\ j = 1,\ldots,k$, the H-function reduces to the G-function.

If we take $R = 1 = \tau_i$ in (39), the unknown density $g(w_1)$ is obtained in terms of H-function . That is

$$g(w_1) = \prod_{j=1}^{k}\frac{A^{\frac{\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j + tm}{\gamma_j} + \beta_j\right)}{\Gamma\left(\frac{\alpha_j + tm}{\gamma_j}\right)}\left\{H_{r,r}^{r,0}\left[\prod_{j=1}^{r} A^{\frac{\delta_j}{\gamma_j}} w_1 \left| \begin{matrix} \left(\frac{\alpha_j - \delta_j}{\gamma_j} + \beta_j, \frac{\delta_j}{\gamma_j}\right) \\ \left(\frac{\alpha_j + tm - \delta_j}{\gamma_j} + \beta_j, \frac{\delta_j}{\gamma_j}\right) \end{matrix}\right.\right]\right\}$$

$$\left\{H_{k-r,k-r}^{0,k-r}\left[\prod_{j=r+1}^{k} A^{\frac{-\delta_j}{\gamma_j}} w_1 \left| \begin{matrix} \left(1 - \frac{\alpha_j + tm + \delta_j}{\gamma_j}, \frac{\delta_j}{\gamma_j}\right) \\ \left(1 - \frac{\alpha_j + \delta_j}{\gamma_j} - \beta_j, \frac{\delta_j}{\gamma_j}\right) \end{matrix}\right.; j = r+1,\ldots,k\right]\right\} \tag{51}$$

For $\gamma_j = 1 = \delta_j;\ j = 1,\ldots,k$, the H-function reduces to the G-function.

If we take $R = 1 = \tau_i$ in (42), the unknown density $g(w_1)$ is obtained in terms of H-function . That is

$$g(w_1) = J_v(p)\prod_{j=1}^{k}\frac{A^{\frac{\delta_j}{\gamma_j}}\Gamma\left(\frac{\alpha_j + tm}{\gamma_j} + \beta_j\right)}{\Gamma\left(\frac{\alpha_j + tm}{\gamma_j}\right)}\left\{H_{r,r}^{r,0}\left[\prod_{j=1}^{r} A^{\frac{\delta_j}{\gamma_j}} w_1 \left| \begin{matrix} \left(\frac{\alpha_j - \delta_j}{\gamma_j} + \beta_j, \frac{\delta_j}{\gamma_j}\right) \\ \left(\frac{\alpha_j + tm - \delta_j}{\gamma_j} + \beta_j, \frac{\delta_j}{\gamma_j}\right) \end{matrix}\right.\right]\right\}$$

$$\left\{H_{k-r,k-r}^{0,k-r}\left[\prod_{j=r+1}^{k} A^{\frac{-\delta_j}{\gamma_j}} w_1 \left| \begin{matrix} \left(1 - \frac{\alpha_j + tm + \delta_j}{\gamma_j}, \frac{\delta_j}{\gamma_j}\right) \\ \left(1 - \frac{\alpha_j + \delta_j}{\gamma_j} - \beta_j, \frac{\delta_j}{\gamma_j}\right) \end{matrix}\right.; j = r+1,\ldots,k\right]\right\} \tag{52}$$

For $\gamma_j = 1 = \delta_j;\ j = 1,\ldots,k$, the H-function reduces to the G-function.

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