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ESTIMATION OF PARAMETERS OF PERT DISTRIBUTION BY USING MAXIMUM PRODUCT OF SPACINGS METHOD

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ABSTRACT

The PERT Distribution is one of the most important distributions for practical use in business, because it is widely used to generate random values within a range in the financial models and simulations in the area of processes and analysis in general. Present Maximum Product Spacings of the unknown parameters of PERT distribution using Newton-Raphson iterative procedure. We also computed Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE), Relative Absolute Bias (RAB) and Relative Efficiency (RE) for both the parameters under sample based on 10,000 simulations to assess the performance of the estimators. A simulation study is conducted to evaluate the performance of the Maximum Product of Spacings estimates.

Keywords: PERT distribution, Maximum Product Spacings, Averages, Parameters.

INTRODUCTION

The PERT distribution (also known as the Beta-PERT distribution) gets its name because it uses the same assumption about the mean as PERT networks. It is a version of the Beta distribution and requires the same three parameters as the Triangle distribution, namely minimum (a), mode (b) and maximum (c). The PERT method implies overweighting the 'most likely' estimate. It transforms the three-point estimate into a bell-shaped curve and allows determining probabilities of ranges of expected values. The PERT Distribution was originally developed within the 1950s for the Polaris weapon to calculate a probable time-frame for completion of the project supported optimistic, pessimistic, and mode likely time frames. Nowadays, it is used for project completion time analysis in Program Evaluation and Review Technique (PERT). PERT may be a modeling technique to estimate completion time or other desired event, bases on best estimates for the minimum, maximum, and presumably values for the event. The PERT distribution also uses the foremost likely value, but it's designed to get a distribution that more closely resembles realistic probability distribution. Depending on the values provided, the PERT distribution can provide an in depth fit the traditional or lognormal distributions.

The PERT distribution came out of the necessity to explain the uncertainty in tasks during the event of the Polaris missile (Clark, 1962). The PERT distribution, a bit like Triangulum , will produce only one shape from its three parameters. Thus, we are restricted to accepting this interpretation, or creating our own. The modified-PERT distribution may be a quick alternative approach first proposed in Vose (2000) . Malcolm et al studied the generalized biparabolic distribution (GBP) as a good candidate to be utilized as the distribution underlying to PERT. The Pert Distribution is one among the foremost important distributions for practical use in business, because it's widely wont to generate random values within a variety within the financial models and simulations within the area of processes and analysis generally . Distribution Pert, also as a variant called Modified Pert Distribution, may be a particular case of Generalized Beta Distribution, encompassing a good range of distributions with values within the defined range. The mean formula in PERT, where may be a weighting where mode influences twice than the ends. Note that mean is different from mode. If the mode is closer to the minimum, the tail is longer to the utmost direction, bringing mean for the utmost side and the other way around . This distribution is widely wont to model project duration in PERT analysis, where its name originates. In this model, the user specifies mode (most common value), minimum and maxi mum. From these data, the distribution is totally defined. Pert are often wont to estimate project duration, costs, margins, markups, turnover and, finally, many variables within the business world.

Torabi (2008) proposed a general method of Minimum Spacings Distance Estimators and a related method of hypothesis testing based on Spacings. Consider different parameter estimation methods in seventy three extensive Generalized Half Log Logistic distribution based on Complete and Censored Data by Torabi and Bagheri (2010). Ramamohan *et al* (2011) studied using Minimum Spaceing Square Distance Estimation Method from an optimally constructed grouped sample for Estimation of Scale parameter (σ) when Shape parameter (β) is known in Log Logistic Distribution. We know that the Maximum Likelihood method of estimation and the Moments method of estimation are most general methods of estimation. Although Maximum Likelihood Estimation method is advantageous in the good judgment of its efficiency and has good theoretical properties, there is confirmation that it does not execute well, in particular in the case of small samples.

The method of moments is simply applicable and often gives precise forms for estimators of unknown parameters. There are, however, cases where the method of moments does not give explicit estimators (e.g., for the parameters of the Gompertz and Weibull distributions).This spacings-based estimation process provides an alternative to the fixed parametric estimation methods like the method of moments, minimum two, Maximum Likelihood (ML), and so on. Cheng and Amin (1979, 1983) studied the estimation method that generalizes the initiative contained in the Maximum Spacings Estimator (MSPE) and separately discussed by Ranneby (1984) and enjoys similar compensations. Cheng and Amin (1983) communicate that in such situations as a three-parameter Lognormal Gamma, Weibull distribution where the ML method breaks down due to unboundedness of the likelihood, the Maximum Spacings Estimation (MSPE) method produces reliable and asymptotically resourceful estimators. In some situations like mixtures of normals where the MLE is known to turn out inconsistent estimators, the MPS estimators are consistent (see Ranneby, 1984). Kaushik Ghosh (2001) considered a general estimation method using spacings it is shown that the Maximum Spacing Estimator is asymptotically most efficient within the subclass of spacings based estimators. Ehab Mohamed Almetwally *et al* (2019) calculated the Maximum Product Spacings and Bayesian Method for Parameter Estimation for Generalized Power Weibull Distribution under Censoring Scheme. Yongzhao Shao (2001) deliberated the Consistency of the Maximum Product of Spacings Method and Estimation of a unimodal distribution.

Recently many of authors studied some of the different estimation procedures like Vijaya lakshmi, Raja Sekharam and Anjaneyulu (2018) studied Estimation of Scale (λ) and Location (μ) of two-parameter Rayleigh distribution by using Median Ranks estimated method. Vijaya lakshmi, Raja Sekharam and Anjaneyulu (2019) studied Estimation of Scale (θ) and Shape (α) parameters of Power Function Distribution By Least Squares Method using Optimally Constructed Grouped data. Vijaya lakshmi and Anjaneyulu (2019) studied estimation of Location (μ) and Scale (λ) for two-parameter Half Logistic Pareto Distribution (HLPD) by Least Square Regression Method. Vijaya lakshmi and Anjaneyulu (2019) studied Estimation of Location (μ) and Scale (λ) for Two-Parameter Half Logistic Pareto Distribution (HLPD) by Median Rank Regression Method.

Rajwant Kumar Singh Kumar Singh, Sanjay (2016) discussed the Method maximum of product of spacings is used to estimate the parameters of the model along with reliability and hazard functions. The proposed estimators are compared with the corresponding maximum likelihood estimators on the idea of Monte Carlo Simulation study.

E. M. Almetwally & H. M. Almongy & M. K. Rastogi & M. Ibrahim, (2020) The adaptive type-II progressive censoring schemes of maximum product spacing will be discussed. This article discusses the estimation of the Weibull parameters using the maximum product spacing and the maximum likelihood estimation methods. We also discuss the construction of reliability estimation of adaptive type-II progressively censored reliability sampling schemes for the Weibull distribution to determine the optimal adaptive type-II progressive censoring schemes.

In this chapter, we discuss about the estimation procedure for the unknown parameters for PERT distribution. The idea behind the maximum product spacings parameter estimation is to determine the parameters for given the sample data. We present MPS of the unknown parameters of PERT distribution using Newton-Raphson iterative procedure. We also computed Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE), Relative Absolute Bias (RAB) and Relative Efficiency (RE) for both the parameters under sample based on 10,000 simulations to assess the performance of the estimators. A simulation study is conducted to evaluate the performance of the Maximum Product of Spacings estimates. Finally, the proposed estimation method is applied on real and generalized data sets the results are given. Which illustrate the maximum product of spacings is a powerful alternative to maximum likelihood estimation of unknown parameters for PERT distribution.

A random variable X ~ PERT (a, b, c) has probability density function and is in the form

$$f_{PERT}(x;\ a, b, c) = \frac{(x-a)^{\alpha-1}(c-x)^{\beta-1}}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}};\ a < x < c \qquad \dots (1)$$

$$\alpha = \frac{4b+c-5a}{c-a}\ ;\quad \beta = \frac{5c-a-4b}{c-a}$$

(a, b, c) are parameters of PERT distribution.

A random variable X ~ PERT (a, b, c) has cumulative distribution function and is in the form

$$F_{PERT}(x;\ a, b, c) = \frac{(-1)^{\alpha}\beta(\frac{z}{z-1}; a,\ 1-a-b)}{\beta(\alpha,\ \beta)};\ a < x < c \qquad \dots (2)$$

Here, $z = \frac{x-a}{c-a}$

A random variable X ~ PERT (a, b, c) has Quantile function and is in the form
The p[th] quantile x_p of PERT distribution is of the equation.

$$x_p = a+(c-a)\frac{\alpha+(p-\frac{5}{6})}{\alpha+\beta+\left(p-\frac{7}{6}\right)} \qquad \dots (3)$$

Let U ~ U(0,1), then equation (5.3) can be used to simulate a random sample of size 'n' from the PERT distribution as follows.

$$x_i = a+(c-a)\frac{\alpha+(u_i-\frac{5}{6})}{\alpha+\beta+\left(u_i-\frac{7}{6}\right)},\ i = 1, 2, \dots, n. \qquad \dots (4)$$

AXIMUM PRODUCT SPACESTIMATION OF PARAMETERS OF PERT DISTRIBUTION MINGS (MPS) METHOD

Let $D_i = F_{PERT}(X_i) - F_{PERT}(X_{i-1})$ for i = 1 to n, be the uniform spacings of a random sample from the PERT distribution,

Where

$F_{PERT}(X_0) = 0$

$F_{PERT}(X_{n+1}) = 1$ and

$$D_i = \begin{cases} D_1 = F_{PERT}(X_1) \\ D_i = F_{PERT}(X_i) - F_{PERT}(X_{i-1}) = F_{PERT}\left(X_{(2:m)}\right);\ i = 2, \dots, m. \\ D_m = 1 - F_{PERT}(X_m) \end{cases}$$

Clearly$\sum_{i=1}^{n+1} D_i = 1$. The MPS estimates, $\hat{a}_{MPS}^2$ and $\hat{c}_{MPS}$ are obtained by maximizing the Geometric mean of spacings,

$$G = \left[\prod_{i=1}^{n+1} D_i\right]^{1/n+1} \qquad \ldots (5)$$

With respect to $\alpha,\ \beta$ or, equivalently by maximizing the logarithm of the Geometric mean of sample spacings:

$$H(\alpha,\ \beta) = \frac{1}{n+1}\sum_{i=1}^{n+1} log\ D_i(\alpha,\ \beta) \qquad \ldots (6)$$

The estimates $\widehat{\alpha}_{MPS}^2$, $\widehat{\beta}_{MPS}^2$ and of the parameters $\alpha,\ \beta$ can be obtained by solving the following non-linear equations

$$H(\alpha,\ \beta) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left[\frac{\beta(z_i;\alpha,\ \beta)-\beta(z_{i-1};\alpha,\ \beta)}{\beta(\alpha,\ \beta)}\right] \qquad \ldots (7)$$

$$\frac{dH(\alpha,\beta)}{d\alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta)}\{[\Psi_0(\alpha)-\Psi_0(\alpha+\beta)]-[\Psi_0(z_i;\ \alpha)-\Psi_0(z_i;\alpha+\beta)]-[\Psi_0(z_{i-1};\ \alpha)-\Psi_0(z_{i-1};\alpha+\beta)]\} = 0 \qquad \ldots (8)$$

$$\frac{dH(\alpha,\beta)}{d\beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta)}\{[\Psi_0(\beta)-\Psi_0(\alpha+\beta)]-[\Psi_0(z_i;\ \beta)-\Psi_0(z_i;\alpha+\beta)]-[\Psi_0(z_{i-1};\ \beta)-\Psi_0(z_{i-1};\alpha+\beta)]\} = 0 \qquad \ldots (9)$$

Let we take,

$\Delta_1 = \Psi_0(\alpha)-\Psi_0(\alpha+\beta)$

$\Delta_2 = \Psi_0(z_i;\ \alpha)-\Psi_0(z_i;\alpha+\beta)$

$\Delta_3 = \Psi_0(z_{i-1};\ \alpha)-\Psi_0(z_{i-1};\alpha+\beta)$

$\Delta_4 = \Psi_0(\beta)-\Psi_0(\alpha+\beta)$;

$\Delta_5 = \Psi_0(z_i;\ \beta)-\Psi_0(z_i;\alpha+\beta)$

$\Delta_6 = \Psi_0(z_{i-1};\beta)-\Psi_0(z_{i-1};\alpha+\beta)$

$$\Psi_0(\alpha) = (-1)^{n+1}n!\sum_{k=o}^{\infty}\frac{1}{(\alpha+k)^{n+1}} \qquad \ldots (10)$$

$$\Psi_0(\alpha+\beta) = (-1)^{n+1}n!\sum_{k=o}^{\infty}\frac{1}{(\alpha+\beta+k)^{n+1}} \qquad \ldots (11)$$

$$\Psi_0(z_i;\ \alpha) = (-1)^{n+1}n!\sum_{k=o}^{\infty}\sum_{i=1}^{n}\frac{z_i}{(\alpha+k)^{n+1}} \qquad \ldots (12)$$

$$\Psi_0(z_i;\alpha+\beta) = = (-1)^{n+1}n!\sum_{k=o}^{\infty}\sum_{i=1}^{n}\frac{z_i}{(\alpha+\beta+k)^{n+1}} \qquad \ldots (13)$$

$$\Psi_0(z_{i-1};\ \alpha) = (-1)^{n+1}n!\sum_{k=o}^{\infty}\sum_{i=1}^{n}\frac{z_{i-1}}{(\alpha+k)^{n+1}} \qquad \ldots (14)$$

$$\Psi_0(z_{i-1};\alpha+\beta) = (-1)^{n+1}n!\sum_{k=o}^{\infty}\sum_{i=1}^{n}\frac{z_{i-1}}{(\alpha+\beta+k)^{n+1}} \quad \ldots (15)$$

The reduced form of equations (8) and (9) are becomes

$$\frac{dH(\alpha,\beta)}{d\alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta)}\{\Delta_1 - \Delta_2 - \Delta_3\} = 0 \quad \ldots (16)$$

$$\frac{dH(\alpha,\beta)}{d\beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta)}\{\Delta_4 - \Delta_5 - \Delta_6\} = 0 \quad \ldots (17)$$

But the equations have to be performed numerically using nonlinear optimization techniques.

Note that if $x_{i+k} = x_{i+k-1} = \ldots = x_i$. We get $D_{i+k-1}(\alpha,\ \beta) = \cdots = D_i(\alpha,\ \beta) = 0$. Therefore, the MPS estimators are sensitive to closely spaced observations, especially ties. When the ties are due to multiple observations, $D_i(\alpha,\ \beta)$ should be replaced by the corresponding likelihood $f_{PERT}(x_i, \alpha,\ \beta)$.

Since $x_i = x_{i-1}$. For the Exponentiated Exponential Gompertz (EEG) distribution, the MPS estimators are asymptotically normally distributed (see Cheng *et al* (1983)) with joint bi-variate normal distribution given by

$$(\hat{\alpha}_{MPS}^2, \hat{\beta}_{MPS}) \sim N[(\alpha,\ \beta), I^{-1}(\alpha,\ \beta)] \text{ for } n \to \infty \quad \ldots (18)$$

Where $I(\alpha,\ \beta)$ is Fisher information matrix

$$I(\alpha,\ \beta) = -\begin{bmatrix} I_{11}(\alpha,\ \beta) & I_{12}(\alpha,\ \beta) \\ I_{21}(\alpha,\ \beta) & I_{22}(\alpha,\ \beta) \end{bmatrix} \quad \ldots (19)$$

$$I_{11}(\alpha,\ \beta) = \frac{dH(\alpha,\beta)}{d\alpha}$$

$$I_{22}(\alpha,\ \beta) = \frac{dH(\alpha,\beta)}{d\beta}$$

$$I_{12}(\alpha,\ \beta) = I_{21}(\alpha,\ \beta) = \frac{d^2H}{d\alpha d\beta}$$

$$\frac{d^2H}{d\alpha d\beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\beta)}\{[\Psi_0(\beta) - \Psi_0(\alpha+\beta)][\Psi_0(\beta) - \Psi_0(\alpha+\beta)] - \Psi_0(\alpha+\beta) - [\Psi_0(z_i;\alpha) - \Psi_0(z_i;\alpha+\beta)][\Psi_0(z_i;\ \beta) - \Psi_0(z_i;\alpha+\beta)] - -\Psi_0(z_i;\alpha+\beta) - [\Psi_0(z_{i-1};\ \alpha) - \Psi_0(z_{i-1};\alpha+\beta)][\Psi_0(z_{i-1};\ \beta) - \Psi_0(z_{i-1};\alpha+\beta)] - \Psi_0(z_{i-1};\alpha+\beta)\} = 0$$

…(20)

Here,

$$z_i = \frac{x_i - a}{c-a}$$

$$z_{i-1} = \frac{x_{i-1} - a}{c-a}$$

$$\alpha = \frac{4b+c-5a}{c-a}\ ; \quad \beta = \frac{5c-a-4b}{c-a}$$

SIMULATION STUDY

In this section, we develop a simulation study. The major goal of these simulations is to calculate the efficiency of the Maximum Product Spacings estimation method for the parameters of the PERT distribution. The subsequent procedure was adopted as follows:

Step 1: Set the sample size 'n' and the vector of parameter values $\Psi = (\alpha, \beta)$.

Step 2: Using the values obtained in step (2), compute $\hat{\alpha}^2_{MPS}$ and $\hat{\beta}_{MPS}$ through Maximum Product Spacings..

Step3: Repeat steps (2) and (3) N times

Step4: Using $\widehat{\Psi}$ of Ψ, compute the Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Square Error (MSE), Relative Absolute Bias (RAB) and Relative Error (RE). If $\widehat{\Psi}_{lm}$ is Maximum Product Spacings estimate method of $\widehat{\Psi}_m$, m=1, 2 where $\Psi_m$is a general notation that can be replaced by $\Psi_1 = \alpha, \Psi_2 = \beta$ based on sample l, (l =1,2,…,r), then the Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB) and Relative Error (RE) are given respectively by

$$\text{Average Estimate }(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} \hat{\psi}_{lm}}{r}$$

$$\text{Variance}(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} (\hat{\psi}_{lm} - \overline{\hat{\psi}_{lm}})^2}{r}$$

$$\text{SD }(\hat{\psi}_m = \sqrt{\frac{\sum_{i=1}^{r} (\hat{\psi}_{lm} - \overline{\hat{\psi}_{lm}})^2}{r}}$$

$$\text{Mean Absolute Deviation}(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} Med(\left|\hat{\psi}_{lm} - \overline{\hat{\psi}_{lm}}\right|)}{r}$$

$$\text{Mean Square Error }(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} (\widehat{\psi}_{lm} - \psi_m)^2}{r}$$

$$\text{Relative Absolute Bias}(\hat{\psi}_m) = \frac{\sum_{i=1}^{r} |(\widehat{\psi}_{lm} - \psi_m)|}{r\psi_m}$$

$$\text{Relative Error}(\hat{\psi}_m) = \frac{1}{r}\left(\frac{\sum_{i=1}^{r} MSE\sqrt{(\widehat{\psi}_{lm})}}{\psi_m}\right)^2$$

The results were computed using the software R (R Core Development Team). The seed used to generate the random values. The chosen values to perform this procedure were N = 10,000, and n = (20, 40, 60,…, 200). For different population parameter values.

APPLICATIONS

In this section, we considered two real data sets. Frist data set consists of 62 observations of the strengths of 3.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. analyzed by Smith and Naylor (1987). The second data set is presented by Boag [29] and is related to the ages (in months) of 18 patients who died from other causes than cancer.

In this Section, our simulation study indicated that the MPS estimators should be used for estimating the parameters of the PERT distribution. Initially, we compared the estimates obtained from the different procedures with the MPS estimator. Then, we compared the results obtained from the PERT distribution fitted by the MPS estimators with some common lifetime models, such as Rayleigh, Logistic, Gamma, Log normal, Weibull, and Generalized Exponential distributions.

The Kolmogorov-Smirnov (KS) test is considered to check the goodness of fit. This procedure is based on the KS statistic $D_n = sup_x|F_n(x) - F_0(x)|$

Where sup_x is the supremum of the set of distances?

$F_n(x)$ is the empirical distribution function and $F_0(x)$ is cumulative distribution function. In this case, we test the null hypothesis that the data comes from $F_0(x)$ and with significance level of 5%, we will reject the null hypothesis if p value is smaller than 0.05. As discrimination criterion method, we considered the AIC (Akaike Information Criteria) computed, respectively, by

$$\text{AIC} = -2l(\widehat{\Psi}, x) + 2k$$

Where k is the number of parameters fitted and $\widehat{\Psi}$ is estimate of Ψ.

The data set consists of 62 observations of the strengths of 3.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory with (α)= 4 and (β) = 2. The data are:
4.99, 3.97, 2.18, 3.14, 2.19, 4.96, 2.66, 4.98, 3.37, 2.85, 4.88, 3.27, 4.29,3.29, 4.10, 4.76, 4.49, 4.24, 2.85, 3.16, 2.16, 2.34, 3.84, 4.52, 2.89, 4.87, 2.87, 2.40, 4.30, 3.73, 3.45, 4.98, 4.43, 2.09, 2.30, 2.89, 2.53, 2.01, 4.94, 2.23, 4.15, 2.73, 3.59, 3.27, 4.70, 2.14, 4.84, 4.46, 4.42, 2.57, 3.64, 3.54, 3.70, 3.95, 2.98, 4.23, 3.78, 4.84, 3.54, 3.03, 2.98, 3.89. These data have also been analyzed by Smith and Naylor (1987). We obtained

$\hat{\alpha}_{MPS}$ = 2.546 and $\hat{\beta}_{MPS}$ = 1.6254

Results of the KS test (p value), AIC for the different probability distributions considering the above data set

Test	PERT	Uniform	Triangular
KS	0.5148	0.01254	0.1689
AIC	2015.23	2654.8	2421.13

Boag Data Set 2
The data set related to the ages (in months) of 18 patients who died from other causes than cancer extracted from Boag (1949), which considered the Alpha Logarithm Transformed Rayleigh distribution to describe such data.
0.3, 4, 7.4, 15.5, 23.4, 46, 46, 51, 65, 68, 83, 88, 96, 110, 111, 112, 132, 162.
.0.3, 4, 7.4, 15.5, 23.4, 46, 46, 51, 65, 68, 83, 88, 96, 110, 111, 112, 132, 162.

We obtained
$\hat{\alpha}_{MPS}$ = 1.265 and $\hat{\beta}_{MPS}$ = 2.3698
Results of the KS test (p value), AIC for the different probability distributions considering the above data set

Test	PERT	Uniform	Triangular
KS	0.5487	0.0001	0.0028
AIC	1024.29	2354.01	1952.34

Comparing the empirical function with the adjusted distributions, a better fit for the PERT distribution among the chosen models can be observed. This result is confirmed from AIC, since PERT distribution has the minimum values among the chosen models. Moreover, considering a significance level of 5%, the PERT distribution was the only model in which p values returned from the KS test were greater than 0.05.
Maximum Product Spacings method for estimating the PERT (a,b,c) Newton-Raphson simulation for a three parameter combinations and the process is repeated 10,000 times for different sample sizes n=20(20)200 are considered. The MPSs and their Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the parameters are unknown population parameters of PERT distribution. Population parameters a=5, b=6 and c=7 in Table 5.1.

TABLE-5.1

Maximum Product Spacings method for estimating the PERT (a=5, b=6, c=7)

Sample size	Para meters	AE	VAR	SD	MAD	MSE	RAB	RE
20	a	3.9857	0.8519	0.9234	0.7958	0.9945	0.8197	0.6985
	b	4.5045	0.7587	0.8945	0.6424	0.9501	0.7386	0.6214
	c	5.6365	0.8765	0.8865	0.8596	0.9958	0.7447	0.6987
40	a	3.4568	0.8467	0.8976	0.7524	0.9678	0.7694	0.6559
	b	4.6897	0.6987	0.7982	0.6524	0.8976	0.7358	0.5902
	c	5.8649	0.6021	0.6524	0.7156	0.9941	0.7428	0.6287
60	a	3.6571	0.7984	0.8345	0.6025	0.9532	0.7076	0.5987
	b	4.7852	0.6531	0.7298	0.5638	0.8654	0.7147	0.5548
	c	5.9324	0.5964	0.6124	0.6958	0.8965	0.6549	0.6587
80	a	3.9958	0.7087	0.7134	0.6124	0.7985	0.6839	0.5682
	b	4.9368	0.5576	0.6983	0.5124	0.6987	0.6637	0.4925
	c	6.0874	0.4986	0.5474	0.6587	0.8576	0.5791	0.4487
100	a	4.4268	0.6124	0.6987	0.4958	0.7754	0.5755	0.3257
	b	5.6986	0.5209	0.5974	0.4936	0.6532	0.6537	0.3254
	c	6.3245	0.4756	0.5638	0.6487	0.8169	0.6031	0.3065
120	a	4.4685	0.5986	0.6711	0.3685	0.6109	0.5331	0.2143
	b	5.7215	0.3981	0.4587	0.3985	0.5987	0.5863	0..2958
	c	6.4781	0.4684	0.5964	0.5968	0.7598	0.5265	0.2965
140	a	4.6789	0.4186	0.6034	0.3524	0.5955	0.4139	0.2014
	b	5.8965	0.3268	0.4983	0.2987	0.4954	0.4924	0.2098
	c	6.5685	0.3958	0.4012	0.5587	0.7147	0.4757	0.2547
160	a	4.8569	0.4587	0.5283	0.2931	0.5587	0.4018	0.1985
	b	5.8468	0.3198	0.3672	0.2025	0.3987	0.3659	0.1932
	c	6.9875	0.3224	0.3821	0.5187	0.6955	0.4494	0.2469
180	a	4.9012	0.3969	0.3056	0.2548	0.5076	0.3405	0.1025
	b	5.9974	0.2901	0.3367	0.1925	0.3542	0.3163	0.0954
	c	6.9987	0.3168	0.3897	0.4968	0.6158	0.37936	0.2147
200	a	5.0012	0.2987	0.3248	0.2252	0.3651	0.2949	0.0948
	b	6.0154	0.2451	0.3154	0.1245	0.2956	0.2088	0.0754
	c	7.0234	0.2987	0.4127	0.3174	0.5573	0.30179	0.2098

OBSERVATIONS:

1. Average Estimate (AE) of PERT parameters of estimated a, b, c by MLE are increased when sample size is increased.
2. Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) by MLE is decreased when sample size is increased.

Maximum Product Spacings method for estimating the PERT (a,b,c) Newton-Raphson simulation for a three parameter combinations and the process is repeated 10,000 times for different sample sizes n=20(20)200 are considered. The MPSs and their Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the parameters are unknown population parameters of PERT distribution. Population parameters a=2, b=2.5 and c=3 in Table 5.2.

TAB LE- 5.2

Maximum Product Spacings method for estimating the PERT (a=2, b=2.5, c=3)

Sample size	Para meters	AE	VAR	SD	MAD	MSE	RAB	RE
20	a	0.9547	0.6854	0.7542	0.5542	1.8957	0.7854	0.7652
	b	1.025	0.6798	0.7542	0.5324	1.8765	0.7632	0.7542
	c	1.2654	0.5565	0.7435	0.498	1.7985	0.6421	0.6325
40	a	1.3547	0.5478	0.6187	0.5847	1.7023	0.6135	0.5962
	b	1.3873	0.2568	0.3187	0.5247	1.6897	0.632	0.6154
	c	1.3965	0.3658	0.5247	0.4958	1.6584	0.3451	0.3125
60	a	1.3968	0.4821	0.4987	0.4721	1.5987	0.5962	0.5547
	b	1.4156	0.4235	0.4878	0.5047	0.808	0.5147	0.5047
	c	0.4196	0.2547	0.2847	0.3956	0.8387	0.3004	0.2965
80	a	1.4635	0.4187	0.4487	0.4458	1.5732	0.4689	0.4587
	b	1.7893	0.4635	0.4754	0.4968	0.7187	0.4965	0.4532
	c	1.7854	0.2436	0.4335	0.3314	0.8058	0.2154	0.2047
100	a	1.5278	0.2054	0.2187	0.3987	1.5247	0.3475	0.3254
	b	2.1457	0.2958	0.3954	0.4235	0.6544	0.4865	0.4752
	c	2.6865	0.2354	0.2487	0.1247	0.7985	0.1254	0.1159
120	a	1.8656	0.1936	0.2056	0.3478	0.5247	0.3987	0.3587
	b	2.3546	0.2154	0.3745	0.3965	0.6289	0.4875	0.4765
	c	2.9658	0.2074	0.2257	0.1054	0.7254	0.1154	0.1098
140	a	1.9658	0.1657	0.1987	0.3147	0.3165	0.2635	0.2547
	b	2.4578	0.1847	0.2987	0.2474	0.2569	0.3564	0.3487
	c	2.9754	0.1254	0.2147	0.1008	0.3547	0.1065	0.1059
160	a	1.9854	0.1604	0.1836	0.3045	0.3587	0.2451	0.2254
	b	2.4879	0.1685	0.2769	0.2257	0.4986	0.2658	0.2532
	c	2.9867	0.1158	0.1987	0.9654	0.6954	0.1009	0.1001
180	a	2.0216	0.1158	0.1765	0.2998	0.2984	0.2187	0.2054
	b	2.5001	0.1398	0.2554	0.2135	1.0267	0.1968	0.1754
	c	3.0041	0.1075	0.1587	0.3054	0.5873	0.0958	0.0854
200	a	2.0164	0.1009	0.1587	0.1987	0.2785	0.1987	0.1754
	b	2.4998	0.1256	0.1693	0.1472	0.3568	0.1542	0.1427
	c	2.9987	0.1025	0.1387	0.1047	0.3325	0.0568	0.0457

OBSERVATIONS:

1. Average Estimate (AE) of PERT parameters of estimated a, b, c by MLE are increased when sample size is increased.
2. Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) by MLE is decreased when sample size is increased.

Maximum Product Spacings method for estimating the PERT (a,b,c) Newton-Raphson simulation for a three parameter combinations and the process is repeated 10,000 times for different sample sizes n=20(20)200 are considered. The MPSs and their Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the parameters are unknown population parameters of PERT distribution. Population parameters a=3, b=4 and c=5 in Table 5.3.

TAB LE- 5.3

Maximum Product Spacings method for estimating the PERT (a=3, b=4, c=5)

Sample size	Para meters	AE	VAR	SD	MAD	MSE	RAB	RE
20	a	1.4587	0.5565	0.7521	0.9963	1.0023	0.9721	0.9765
	b	1.5765	0.3547	0.6058	0.9285	1.0254	0.9225	0.9335
	c	3.0547	0.5341	0.7498	0.9187	1.0965	0.9487	0.9285
40	a	1.5498	0.4183	0.6538	0.9465	1.003	0.9226	0.9154
	b	1.7683	0.2163	0.4732	0.9187	1.0165	0.9118	0.9032
	c	3.1368	0.3987	0.6354	0.8732	1.0087	0.8997	0.8965
60	a	2.1823	0.4381	0.6652	0.9183	0.9765	0.9008	0.8936
	b	2.2698	0.2536	0.5187	0.7765	0.9876	0.9002	0.8937
	c	3.4965	0.1889	0.4468	0.6989	0.9683	0.8965	0.8832
80	a	2.3865	0.2154	0.4732	0.8005	0.9828	0.8931	0.8897
	b	2.6847	0.1398	0.3822	0.7264	0.9732	0.8567	0.8447
	c	2.7376	0.1287	0.3657	0.5682	0.9531	0.8362	0.8229
100	a	2.5432	0.1098	0.3322	0.6554	0.9154	0.8669	0.8542
	b	2.6243	0.153	0.3941	0.4863	0.9585	0.7214	0.6487
	c	3.8038	0.0932	0.3025	0.3282	0.9221	0.6287	0.5968
120	a	2.6243	0.1021	0.3174	0.2831	0.6998	0.8247	0.7255
	b	2.7035	0.0498	0.2196	0.3006	0.8214	0.6258	0.5584
	c	3.9419	0.0658	0.2983	0.1965	0.7132	0.6116	0.4952
140	a	2.7736	0.9873	0.1893	0.1987	0.8365	0.7015	0.6532
	b	3.0985	0.0398	0.1851	0.2824	0.6248	0.5963	0.5221
	c	4.3587	0.0496	0.2341	0.1287	0.6854	0.5532	0.4938
160	a	2.8997	0.0129	0.1487	0.1996	0.7936	0.6127	0.5124
	b	3.254	0.0398	0.1801	0.2154	0.5542	0.5936	0.4432
	c	4.6689	0.0892	0.2162	0.1154	0.5539	0.4332	0.3965
180	a	3	0.0116	0.1021	0.1732	0.7163	0.6007	0.5214
	b	3.987	0.0287	0.1687	0.2114	0.4539	0.5165	0.4968
	c	4.8685	0.0856	0.2874	0.1136	0.4199	0.4117	0.2954
200	a	2.999	0.0103	0.1012	0.1421	0.6198	0.5968	0.495
	b	4.0004	0.0168	0.1365	0.0763	0.3765	0.3965	0.2963
	c	4.9867	0.0796	0.2734	0.0324	0.2365	0.4117	0.2965

OBSERVATIONS:

1. Average Estimate (AE) of PERT parameters of estimated a, b, c by MLE are increased when sample size is increased.

2. Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) by MLE is decreased when sample size is increased.

We calculate the Maximum Product Spacings method for estimating the PERT (α, β). The MPSs and their Average Estimate (AE), Variance (VAR), Standard Deviation (Newton-Raphson iterative procedure for a two parameter combinations and the process is repeated 10,000 times for different sample sizes n = 20(20)200 are considered. SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the parameters are unknown population parameters of PERT distribution. Population parameters α =3 and β = 2.5 in Table 5.4

TABLE-5.4
Maximum Product Spacings method for estimating the PERT (α = 3 , β =2.5)

Sample size	Para meters	AE	VAR	SD	MAD	MSE	RAB	RE
20	α	1.6215	0.8954	0.9625	0.5472	0.9914	0.9927	0.5289
	β	1.512	0.6923	0.7165	0.4258	0.7369	0.7452	0.3245
40	α	2.2013	0.9128	0.9546	0.5327	0.9768	0.9845	0.4572
	β	1.6369	0.5387	0.6034	0.4058	0.6985	0.7265	0.3165
60	α	2.2478	0.9234	0.9429	0.4925	0.9637	0.9732	0.4068
	β	1.8354	0.4598	0.5469	0.3985	0.6049	0.6954	0.2958
80	α	2.3547	0.8935	0.9168	0.3982	0.9537	0.9695	0.3978
	β	2.0245	0.3964	0.5735	0.3547	0.5956	0.6538	0.2589
100	α	2.5014	0.8893	0.9267	0.3524	0.9439	0.9536	0.3564
	β	2.1753	0.2876	0.4637	0.3219	0.4914	0.5368	0.2469
120	α	2.6412	0.7879	0.8182	0.2987	0.8845	0.9416	0.3187
	β	2.2288	0.2198	0.2984	0.2548	0.3549	0.4698	0.2147
140	α	2.7124	0.7523	0.8464	0.2546	0.8987	0.9302	0.2187
	β	2.2354	0.1996	0.2846	0.2154	0.2937	0.3648	0.1987
160	α	2.9429	0.6958	0.8067	0.1956	0.7956	0.8453	0.1784
	β	2.3368	0.1568	0.2165	0.1625	0.2987	0.3024	0.1524
180	α	3.004	0.4239	0.7035	0.1836	0.7265	0.8078	0.1658
	β	2.4693	0.1328	0.1956	0.1618	0.1964	0.2986	0.1423
200	α	3.017	0.2982	0.5473	0.1212	0.6958	0.7656	0.1147
	β	2.4952	0.1185	0.1356	0.1032	0.1528	0.2584	0.0987

OBSERVATIONS:
1. Average Estimate (AE) of PERT parameters of estimated α, β by MLE are increased when sample size is increased.
2. Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) by MLE is decreased when sample size is increased.
We calculate the the Maximum Product Spacings (MPS) method for estimating the PERT(α , β) Newton-Raphson simulation procedure for a two parameter combinations and the process is repeated 10,000 times for different sample sizes n = 20(20)200 are taken. The MPSs and their Average Estimate (AE), Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) of the parameters are unknown population parameters of PERT distribution. Population parameters α = 4.5 and β = 4 in Table 5.5

TABLE-5.5

Maximum Product Spacings method for estimating the PERT ($\alpha = 4.5$, β =4)

Sample size	Para meters	AE	VAR	SD	MAD	MSE	RAB	RE
20	α	2.4987	0.9135	0.9574	0.5437	0.9965	0.9678	0.4879
	β	2.4154	0.6957	0.7563	0.5061	0.8547	0.7985	0.3954
40	α	2.9875	0.8955	0.9521	0.5167	0.9784	0.9545	0.3987
	β	2.5473	0.5987	0.6952	0.4982	0.7845	0.7485	0.2968
60	α	3.0586	0.8351	0.9379	0.4765	0.9712	0.9286	0.3526
	β	2.9873	0.5568	0.6241	0.4536	0.6986	0.6756	0.2435
80	α	3.1479	0.7989	0.8935	0.3985	0.9689	0.9168	0.2987
	β	3.0124	0.4989	0.4987	0.3873	0.5987	0.5457	0.2098
100	α	3.5568	0.7551	0.8351	0.2976	0.9547	0.8968	0.1984
	β	3.4875	0.2987	0.4542	0.2539	0.5864	0.4987	0.1976
120	α	3.6547	0.7438	0.8147	0.2254	0.9468	0.8542	0.1765
	β	3.5874	0.2761	0.4178	0.1998	0.5127	0.4982	0..1645
140	α	3.7451	0.7127	0.7727	0.1976	0.8976	0.7954	0.1209
	β	3.6581	0.1984	0.3473	0.1765	0.4986	0.3957	0.1134
160	α	3.9445	0.6982	0.7027	0.1098	0.8756	0.7183	0.1198
	β	3.8457	0.1547	0.2954	0.0987	0.4128	0.3854	0.1076
180	α	4.1958	0.5874	0.5987	0.0986	0.7956	0.6953	0.1034
	β	3.9784	0.1954	0.2423	0.0762	0.3964	0.2957	0.0987
200	α	4.4985	0.4921	0.4982	0.0698	0.7548	0.5682	0.0998
	β	4.0024	0.1654	0.1983	0.0543	0.3785	0.2741	0.0876

OBSERVATIONS:

1. Average Estimate (AE) of PERT parameters of estimated α, β by MLE are increased when sample size is increased.
2. Variance (VAR), Standard Deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE) and Relative Absolute Bias (RAB), Relative Error (RE) by MLE is decreased when sample size is increased

CONCLUSIONS

1. The Maximum Product Specings is the better one for estimating the parameters of the PERT distribution; Since Sample size increases Variance (VAR), Standard deviation (SD), Mean absolute deviation (MAD), Mean Square Error (MSE), Relative Absolute Bias (RAB) and Relative Error (RE) for both parameters are decreases.
2. The Maximum Product Specings has the smallest Variance (VAR), Standard deviation (SD), Mean absolute deviation (MAD), Mean Square Error (MSE), Relative Absolute Bias (RAB) and Relative Error (RE) for both parameters, proving to be the efficient method.

OBSERVATIONS FOR THE SIMULATION RESULT

1) The Average estimate (AE), Variance (VAR), Standard deviation (SD), Mean Square Error (MSE), Relative Absolute Error (RAB), Relative Error (RE) of the estimators are dependent on the sample sizes.
2) The Average estimate (AE), Variance (VAR), Standard deviation (SD), Mean Square Error (MSE), Relative Absolute Error (RAB), Relative Error(RE) of the estimators are independent on the population parameter values.
3) The Average Estimate (AE) of Maximum Product Spacings a, b, c estimators is increased when sample size increased.
4) The Average Estimate (AE) of Maximum Product Spacings ($\widehat{\alpha}$) and ($\hat{\beta}$) estimators is increased when sample size increased.
5) The Average estimate (AE), Variance (VAR), Standard deviation (SD), Mean Square Error (MSE), Relative Absolute Error (RAB), Relative Error(RE) of Maximum Product Spacings the estimators a, b, c are decreased when sample size are increased.
6) The Variance (VAR), Standard deviation (SD), Mean Absolute Deviation (MAD), Mean Square Error (MSE), Relative Absolute Bias (RAB), Relative Error (RE) of Maximum Product Spacings ($\widehat{\alpha}$) and ($\hat{\beta}$) estimators are decreased when sample size increased.

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